

Kreweras's Narayana Number Identity Has a Simple Dyck Path Interpretation

DAVID CALLAN

Department of Statistics
University of Wisconsin-Madison
Medical Science Center
1300 University Ave
Madison, WI 53706-1532
callan@stat.wisc.edu

Oct 28, 2005

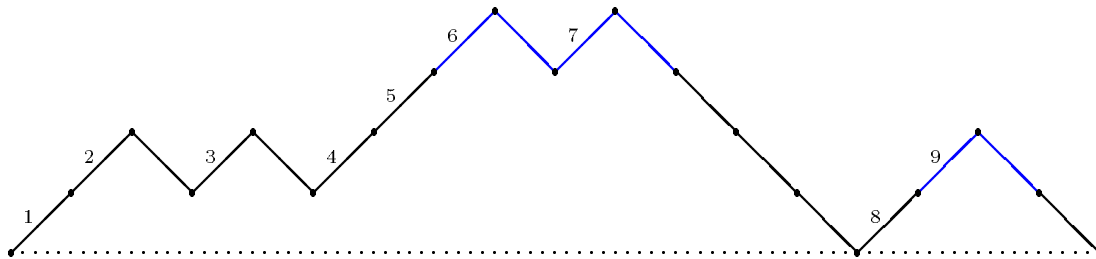
Germain Kreweras [1, 2] proved that

$$\frac{1}{n+r+1} \binom{n+r+1}{r} \binom{n+r+1}{n} = \sum_{s=0}^r \frac{1}{n} \binom{n}{r-s} \binom{n}{r-s+1} \binom{2n+s}{2n}.$$

Replacing r by $r-1$, this identity can be rewritten as

$$N(n+r, r) = \sum_{s=1}^r N(n, s) \binom{2n+r-s}{r-s} \tag{*}$$

where $N(n, k) := \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ is the Narayana number, well known to count Dyck n -paths containing k peaks [3, Ex. 6.36, p. 237]. A Dyck n -path is a path of n upsteps U and n downsteps D that never dips below the line joining its start and end points. A peak is an occurrence of UD . We will show that (*) counts Dyck $(n+r)$ -paths with r peaks by number s of peak plateaus, where a peak plateau is a run of consecutive peaks that is preceded by an upstep and followed by a downstep, that is, a subpath of the form $U(UD)^i D$ with $i \geq 1$. For example, the path illustrated below has 5 peaks but only 2 peak plateaus (in blue, following upsteps 5 and 8 respectively).



Dyck path with 2 peak plateaus

Given a Dyck $(n+r)$ -path with r peaks, delete all its peaks. The result is a Dyck n -path with s peaks where s is the number of peak plateaus in the original path. Each Dyck n -path with s peaks is hit $\binom{2n+r-s}{r-s}$ times: the preimages are found by inserting $r-s$ UD s, repetition allowed, among the $2n+1$ vertices of the n -path— $\binom{2n+r-s}{r-s}$ choices—and then inserting a UD at each of the s original peak vertices, and (*) follows.

References

- [1] G. Kreweras, Traitement simultané du “Problème de Young” et du “Problème de Simon Newcomb”, *Cahiers du Bur. Univ. de Rech. Opér.* **10**, 1967, 23–31.
- [2] T. V. Narayana, *Lattice Path Combinatorics With Statistical Applications*, Mathematical Expositions No.23, Univ. of Toronto Press, 1979.
- [3] Richard P. Stanley, *Enumerative Combinatorics* Vol.2, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at <http://www-math.mit.edu/~rstan/ec/> .