# Lagrange Inversion and Schröder Trees

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1210 W. Dayton St
Madison, WI 53706-1693
**callan@stat.wisc.edu**

A *compositional inverse* of a formal power series $F(x) = \sum_{n\geq 0} a_n x^n$ is a power series $f(x)$ such that $F\big(f(x)\big) = f\big(F(x)\big) = x$. The nonconstant power series $F$ possessing a compositional inverse are precisely those with zero constant term and nonzero $x$ term ($a_0 = 0$, $a_1 \neq 0$) and then the compositional inverse is unique, denoted $F^{\langle -1\rangle}$. Thus $(x+x^2+x^3+\dots)^{\langle -1\rangle} = x-x^2+x^3-\dots$ as is easily verified since the series are, respectively, $x/(1-x)$ and $x/(1+x)$. A power series $F$ with $a_0 = 0$, $a_1 \neq 0$ can be expressed as $F(x) = x/G(x)$ where $G(x) = \sum_{n\geq 0} t_n x^n$ is a power series with nonzero constant term. The celebrated Lagrange inversion formula gives the coefficients of $F^{\langle -1\rangle}(x)$, in fact of $F^{\langle -1\rangle}(x)^k$ (and thus of $P\big(F^{\langle -1\rangle}(x)\big)$ for any power series $P$), in terms of the coefficients of $G$. We use $[x^n]f(x)$ to denote the coefficient of $x^n$ in $f(x)$. The notation $1^{r_1}2^{r_2}\dots n^{r_n} \vdash n$ means that $1^{r_1}2^{r_2}\dots n^{r_n}$ is a partition of $n$ with $r_1$ 1's, $r_2$ 2's, etc. Thus $\sum_i i r_i = n$ and $\sum_i r_i$ is the number of parts.

**Theorem (Lagrange Inversion)** With notation as above, $[x^n]F^{\langle -1\rangle}(x)^k =$

$$\frac{k}{n}[x^{n-k}]G(x)^n \qquad \text{(compact form)} \qquad (1)$$

$$\frac{k}{n} \sum_{1^{r_1}\dots(n-k)^{r_{n-k}} \vdash n-k} \binom{n}{n-\sum_i r_i, r_1, \dots r_{n-k}} t_0^{n-\sum_i r_i} t_1^{r_1} \dots t_{n-k}^{r_{n-k}} \qquad \text{(explicit form)} \qquad (2)$$

The compact and explicit forms are equivalent *via* the multinomial theorem. Incidentally, if you were to to equate coefficients in $f\big(x/G(x)\big) = x$ and start solving the resulting equations, it would soon become clear that all the numerical coefficients in $F^{\langle -1\rangle}(x)$ are integral, a fact not immediately obvious from the formulas (1), (2). Contrariwise, the formulas show that the coefficients in $F^{\langle -1\rangle}(x)$ are all positive, which is not immediately obvious from the equations.

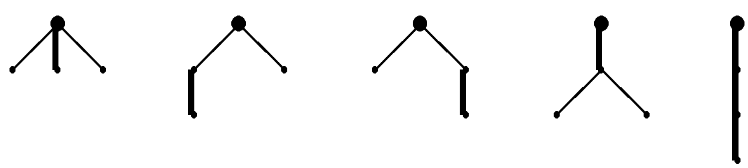
But what about expressing the coefficients of $F^{\langle -1\rangle}(x)^k$ directly in terms of the coefficients of $F(x)$ itself (rather than of $G(x) = x/F(x)$)? The purpose of this note is to point out that there is such a formula, similar to (2) in that it involves a sum over all partitions of $n-k$ but differing in that the coefficients are not all positive.

**Theorem (Direct Inversion)** Let $F(x) = \sum_{n\geq 1} a_n x^n$ where the $a_n$ are commuting indeterminates. Then $[x^n]F^{\langle -1\rangle}(x)^k =$

$$k \sum_{1^{r_1}\ldots(n-k)^{r_{n-k}}\vdash n-k} \frac{(-1)^{\Sigma_i r_i}}{n+\sum_i r_i}\binom{n+r_1+r_2+\ldots+r_{n-k}}{n, r_1, r_2, \ldots, r_{n-k}} a_1^{-(n+\Sigma_i r_i)} a_2^{r_1} a_3^{r_2} \ldots a_{n-k+1}^{r_{n-k}} \qquad (3)$$

To prove this, we adapt the pretty combinatorial proof of (2) due to Raney given in [1].

A *plane tree* is a finite nonempty unlabeled rooted ordered tree. The 5 plane trees with 4 vertices are pictured below and in fact the Catalan number $C_{n-1}$ gives the number of plane trees with $n$ vertices.



(The second and third trees would coincide if they weren't ordered trees and the fifth would too if they weren't rooted.) The *degree* of a vertex is the number of its successors, and the *degree sequence* of a tree is $0^{r_0}1^{r_1}2^{r_2}\ldots$ where $r_i$ is the number of vertices of degree $i$. Thus the degree sequence of the first tree depicted above is $0^3 1^0 2^0 3^1$. Now let $\mathcal{T}$ denote the set of *all* plane trees. The idea in Raney's proof (though he expressed it differently) is to assign to each tree an appropriate size (positive integer) and weight (monomial in the coefficients $t_i$ of $G(x)$), define

$$f(x) = \sum_{\tau\in\mathcal{T}} \text{weight}(\tau)x^{\text{size}(\tau)}$$

and then show that

$$xG(f(x)) = f(x).$$

Since $G(x) = x/F(x)$, this is equivalent to $x = F\big(f(x)\big)$ and the rôle of the esoteric series $f$ is revealed: it is the compositional inverse of $F$. The appropriate size for $\tau \in \mathcal{T}$ is simply the number of vertices in $\tau$ and the appropriate weight is $t_0^{r_0} t_1^{r_1} t_2^{r_2} \ldots$ where the $r_i$ give the degree sequence of $\tau$. Equivalently, weight each vertex of degree $i$ with $t_i$ and multiply these weights over all vertices. Next, an ingenious representation of plane trees as lattice paths *via* their degree sequences permits counting the number of plane trees with a given degree sequence [1, Theorem 5.3.10]. In fact it permits counting plane forests (= ordered sequence of plane trees) and this takes care of raising $f(x)$ to the $k$th power.

The same method works to establish (3) except we use a different family of trees—Schröder trees—with a different measure of size and weight. A *Schröder tree* is a plane tree in which no vertex has degree 1; thus the tree forks at each interior (non-leaf) vertex. Let $\mathcal{S}$ denote the set of all Schröder trees. The size of $\tau \in \mathcal{S}$ is the number of leaf vertices in $\tau$. Note this measure of size forces some such restriction as Schröder trees; otherwise there'd be infinitely many trees of each size. The weight of $\tau \in \mathcal{S}$ is $(-1)^{\#\,\text{interior vertices}} a_1^{-\#\,\text{vertices}} a_2^{r_2} \ldots a_n^{r_n}$ where $0^{\#\,\text{leaf vertices}} 2^{r_2} \ldots n^{r_n}$ is the degree sequence of $\tau$. (By definition, $r_1 = 0$.) We note that the number of Schröder trees of size $n$ is given by the (little) Schröder number $s_n$, A080243 in the online EIS.

Now let $f(x) = \sum_{\tau \in \mathcal{S}} \text{weight}(\tau) x^{\text{size}(\tau)}$. We wish to show that $F\big(f(x)\big) = x$. The crucial insight is that by removing the root and its incident edges from a Schröder tree $\tau$ of size $> 1$ whose root has degree $j$ (necessarily $\geq 2$), we get an ordered sequence of $j$ smaller Schröder trees whose sizes add up to size$(\tau)$ and whose weights multiply to $-a_1\,\text{weight}(\tau)/a_j$. Furthermore, the solitary Schröder tree of size 1—the root is the only vertex—has weight $a_1^{-1}$. Hence for $j \geq 2$,

$$\begin{aligned} a_j f(x)^j &= a_j \sum_{\tau_1 \in \mathcal{S}, \ldots, \tau_j \in \mathcal{S}} \text{weight}(\tau_1) \ldots \text{weight}(\tau_j) x^{\text{size}(\tau_1) + \ldots + \text{size}(\tau_j)} \\ &= -a_1 \sum_{\tau \in \mathcal{S}:\ \deg(\text{root}) = j} \text{weight}(\tau) x^{\text{size}(\tau)} \end{aligned}$$

Summing over all $j \geq 2$,

$$\sum_{j\geq 2} a_j f(x)^j = -a_1\big(f(x) - \frac{1}{a_1}x\big) = x - a_1 f(x)$$

and the desired equality $F\big(f(x)\big) = x$ follows. Thus $f = F^{\langle -1\rangle}$. Now

$$\begin{aligned} f(x)^k &= \sum_{\tau_1\in\mathcal{S},\dots,\tau_k\in\mathcal{S}} \text{weight}(\tau_1)\dots\text{weight}(\tau_k)x^{\text{size}(\tau_1)+\dots+\text{size}(\tau_k)} \\ &= \sum_{\sigma\in\mathcal{S}_{n,k}} \text{weight}(\sigma)x^n \end{aligned}$$

where $\mathcal{S}_{n,k}$ denotes the set of (ordered) Schröder forests with $k$ components and $n$ leaf vertices, and the definitions of size and weight carry over from trees to forests. The degree sequence of $\sigma \in \mathcal{S}_{n,k}$ is necessarily of the form $0^n 2^{q_2}\dots(n-k+1)^{q_{n-k+1}}$ for some sequence $(q_i)$. The total number of edges is then $\sum_{i\geq 2} iq_i$, of interior vertices is $\sum_i q_i$, and of leaf vertices is $n$. In any forest, # vertices = # edges − # components; hence, setting $r_i = q_{i+1}$ $(1 \leq i \leq n-k)$, we find $\sum_i ir_i = n-k$ and $1^{r_1}2^{r_2}\dots(n-k)^{r_{n-k}} \vdash n-k$. Conversely, each partition of $n-k$ gives rise to a possible degree sequence.

It only remains to count forests in $\mathcal{S}_{n,k}$ with degree sequence $0^n 2^{r_1}\dots(n-k+1)^{r_{n-k}}$ corresponding to a partition $1^{r_1}2^{r_2}\dots(n-k)^{r_{n-k}}$ of $n-k$. To do so, we use Theorem 5.3.10 from [1, p. 34].

**Theorem (Counting Forests by Degree Sequence)** The number of plane forests with degree sequence $0^{q_0}1^{q_1}\dots m^{q_m}$ is

$$\frac{k}{\sum_i q_i}\binom{q_0+q_1+\dots+q_m}{q_0,q_1,\dots,q_m}$$

where $k = \sum_i q_i - \sum_i iq_i$ is the number of components.

Applying this result, the required count turns out to be $\frac{k}{n+\sum_i r_i}\binom{n+r_1+r_2+\dots+r_{n-k}}{n,r_1,r_2,\dots,r_{n-k}}$. We are done.

## References


[1] Richard P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 1999.