The 136th Manifestation of C_n

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Abstract

We show bijectively that the Catalan number C_n counts Dyck $(n+1)$-paths in which the terminal descent is of even length and all other descents to ground level (if any) are of odd length.

Richard Stanley's inventory of combinatorial interpretations of the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ currently stands at 135 items. Here is one more.

Theorem 1. *Let $\mathcal{A}_n$ denote the set of Dyck n-paths for which the terminal descent is of even length and all other descents to ground level (if any) are of odd length. Then $|\mathcal{A}_n| = C_{n-1}$ for $n \geq 2$.*

This result is a counterpart to item (j) in Stanley's inventory, which says that C_{n-1} also counts Dyck n-paths for which *all* descents to ground level are of odd length.

A Dyck n-path is a lattice path of n upsteps U and n downsteps D that never dips below *ground level*, the horizontal line joining its start and end points. The number of Dyck n-paths is well known to be C_n. The *size*, also called the *semilength*, of a Dyck n-path is n. A *return* is a downstep that returns the path to ground level. A *descent* is a maximal sequence of contiguous downsteps. A *peak* is an occurrence of UD. A *low* peak (resp. low UDU) is one that starts at ground level. A low peak is also called a *hill* and a low UDU an *early hill*. Note that a path free of early hills is either hill-free or has just one hill at the very end. Hill-free Dyck paths and Dyck paths with an even-length terminal descent are both counted [1] by the Fine numbers, A000957 in OEIS. Early-hill-free Dyck paths are counted [2] by A000958 .

We prove the following refinement of $|\mathcal{A}_n| = C_{n-1}$.

Theorem 2. *For $n \geq 2$ and $k \geq 1$, the paths in $\mathcal{A}_n$ with k returns correspond bijectively to Dyck $(n-1)$-paths that contain $k-1$ early hills.*

The proof relies on the following bijections.

Proposition 3. *There exists a bijection from Dyck n-paths with terminal descent of even (resp. odd) length to hill-free (resp. early-hill-free) Dyck n-paths.*

Proof The "*DUtoDXD*" bijection of [3, §4] establishes the even-length terminal descent $\rightarrow$ hill-free part. For the odd-length terminal descent $\rightarrow$ early-hill-free part, split the first set of paths into A: those with only one return, and B: those with 2 or more returns. The interior (drop first and last steps) of a path in A has terminal descent of even length and so corresponds to a hill-free Dyck $(n-1)$-path by the previous part. Append UD to get a bijection from A to the early-hill-free Dyck n-paths that end UD. A path in B can be written (uniquely) as $PUQD = P \diagup \overset{Q}{} \diagdown$ where P, Q are nonempty Dyck paths and Q has terminal descent of even length. Map to $\diagup \overset{P}{} \diagdown Q'$, where Q' is the hill-free path corresponding to Q. This gives a bijection from B to the early-hill-free Dyck n-paths that do not end UD. $\square$

Proof of Theorem 2 Given a path in $\mathcal{A}_n$ with k returns, use the path's returns to write it (uniquely) as $\diagup \overset{P_1}{} \diagdown \diagup \overset{P_2}{} \diagdown \ \dots \ \diagup \overset{P_{k-1}}{} \diagdown \diagup \overset{P_k}{} \diagdown$ where $P_1, P_2, \dots, P_{k-1}$ are Dyck paths, all with terminal descent of even length (possibly 0), and P_k is a Dyck path with terminal descent of odd length. Using Prop. 3, map the path to $P_1' \diagup\diagdown P_2' \diagup\diagdown \ \dots \ \diagup\diagdown P_{k-1}' \diagup\diagdown P_k'$, where P_i' is hill-free for $1 \leq i \leq k-1$ and P_k' is nonempty early-hill-free. The resulting Dyck path has one fewer U and D than the original and contains $k-1$ early hills, and Theorem 2 follows. $\square$

These results can be used to explain the distribution of the statistic "# even-length descents to ground level" on Dyck paths. First, let $T(n,k)$ denote the number of Dyck n-paths with k returns; $\big(T(n,k)\big)_{0 \leq k \leq n}$ forms the Catalan triangle, A106566 in OEIS.

Corollary 4 ([4]). *The number of Dyck n-paths with k even-length descents to ground level is $T(n, 2k) + T(n, 2k+1)$.*

Proof Again calling on the "*DUtoDXD*" bijection of [3, §4], it sends Dyck n-paths all of whose returns to ground level have even length to Dyck n-paths that start UD and thence (transfer this D to the end of the path) to Dyck n-paths with exactly 1 return.

This establishes the case $k = 0$. For $k \geq 1$, split the paths into A: those for which the terminal descent has even length, and B: the rest. A path in A splits, via its even-length descents to ground level, into k Dyck paths to each of which Theorem 1 applies. The result is a k-list of *nonempty* Dyck paths of total size $n - k$. Since nonempty Dyck paths correspond to 2-return Dyck paths of size 1 unit larger $(\diagup^{P}\diagdown Q \rightarrow \diagup^{P}\diagdown\diagup^{Q}\diagdown)$, we get a bijection from A to Dyck n-paths with $2k$ returns. There is a similar bijection from B to Dyck n-paths with $2k + 1$ returns. $\square$

References

[1] Emeric Deutsch and Louis Shapiro, A survey of the Fine numbers, *Disc. Math.*, **241**, Issue 1-3 (October 2001), 241–265.

[2] Yidong Sun, The statistic "number of udu's" in Dyck paths, *Disc. Math.*, **287** (2004), Issue 1-3 (October 2004), 177-186.

[3] David Callan, Some identities for the Catalan and Fine numbers, preprint, 2005, http://front.math.ucdavis.edu/math.CO/0507169

[4] Yidong Sun, Identities involving some numbers related to Dyck paths, preprint, 2005.