NOTES ON MOTZKIN AND SCHRÖDER NUMBERS

David Callan

Department of Statistics
University of Wisconsin-Madison
1210 W. Dayton Street
Madison, WI 53706-1693
callan@stat.wisc.edu

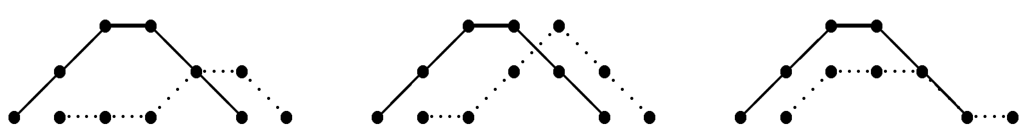
1. Motzkin numbers

The Motzkin number M_n is the cardinality of the set of sequences $\mathcal{M}_n = \{(x_i)_{i=1}^n : x_i \in \{-1, 0, 1\}$, all partial sums $\sum_{i=1}^k x_i$ are nonnegative and $\sum_{i=1}^n x_i = 0\}$. These sequences have a pictorial representation as *Motzkin paths*: lattice paths of upsteps (corresponding to $+1$), downsteps (-1) and flatsteps (0) that begin and end at, but never dip below, "ground level". For example, the $M_3 = 4$ Motzkin 3-paths are pictured in Figure 1.

Figure 1

If flatsteps are disallowed, the resulting paths, known as Dyck paths ("mountain ranges"), are counted by the Catalan number C_n [2,4] where n now denotes the number of upsteps (= number of downsteps).

The sequence of Motzkin numbers $(M_n)_{n\geq 0} = \{1, 1, 2, 4, 9, 21, 51, \dots\}$ is logarithmically convex, that is, $M_n^2 \leq M_{n-1}M_{n+1}$, $n \geq 1$ [1]. Our first note is a combinatorial proof of that fact: an injection $\mathcal{M}_n \times \mathcal{M}_n \to \mathcal{M}_{n-1} \times \mathcal{M}_{n+1}$. Start with a pair of Motzkin n-paths, the second placed so that it begins one unit to the right of the first. Scan the paths left to right and locate the first "close encounter" defined as a point of intersection—either at a lattice point as in Figure 2a or at the center point of crossing diagonal steps as in Figure 2b—or a pair of flatsteps forming the top and bottom of a unit square as in Figure 2c.

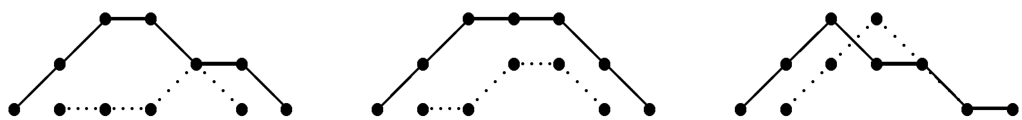


a b c

Figure 2

Typeset by $\mathcal{AMS}$-TeX

Certainly at least one such close encounter exists. In the situation of Figure 2a, reassign the two initial segments to the other path. In that of Figure 2b, swing the crossing steps 45° so they become horizontal. In that of Figure 2c, change the lower horizontal step to an upstep and the upper one to a downstep. The pairs in Figure 2 thus yield respectively the pairs in Figure 3.



a b c

Figure 3

In all cases, the result will be a pair of paths, one in $\mathcal{M}_{n-1}$, the other in $\mathcal{M}_{n+1}$. Furthermore, the location of the first close encounter will remain invariant; so the mapping is reversible and hence an injection. The only elements of $\mathcal{M}_{n-1} \times \mathcal{M}_{n+1}$ not hit will be the pairs of nonintersecting paths having no flatsteps as sides of a unit square. Such pairs exist except for $n = 2$; this proves the desired inequality with equality only for $n = 2$.

2. Bijection from bushes to Motzkin paths

Just as for the Catalan numbers, there are numerous combinatorial manifestations of the Motzkin numbers besides lattice paths (see [3] for a comprehensive survey). We'll consider one related to trees. First we recall that there are several easily-grasped bijections from tree-like structures to lattice paths that involve "walking around the tree". For example, rooted ordered trees with n edges correspond to Dyck paths of n upsteps (and n downsteps) as follows. Draw the tree up from the root. Then walk around the tree clockwise starting at the root, closely following the edges. Thus each edge gets traversed twice in opposite directions. Simply record an upstep when you travel up an edge and a downstep when you travel down an edge. As another example, full binary trees with n interior vertices (and hence $2n$ edges and $n+1$ leaf vertices) also correspond to Dyck paths of n upsteps. Again walk around the tree but this time processing in turn each edge *that has not previously been traversed*: a left-leaning edge becomes an upstep and a right-leaning edge becomes a downstep. Note that in the former case each edge corresponds to *two* steps in the lattice path—an upstep *and* a downstep. In the latter case, each edge corresponds to only one step and we rely on the "full binary" property of the tree to ensure equal numbers of upsteps and downsteps.

To return to Motzkin numbers, a *bush* or *branch-reduced tree* is a rooted, ordered tree in which only the root is allowed exactly one child; in other words, each non-root vertex is either a leaf or has at least 2 children. The 4 bushes with 4 edges are shown in Figure 4 (drawn downward).

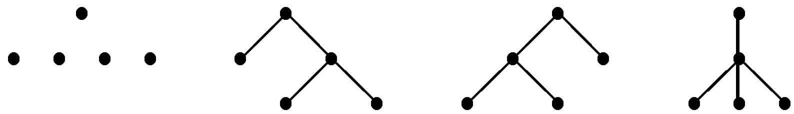


Figure 4

Let $\mathcal{B}_n$ denote the set of bushes with n edges. It is shown in [3] that $|\mathcal{B}_n| = M_{n-1}$. For our second note, we will confirm this by exhibiting another walk-around bijection, from $\mathcal{B}_n$ to $\mathcal{M}_{n-1}$, that incorporates features of both the preceding bijections. Start with a bush in $\mathcal{B}_n$. Walk around it counterclockwise as usual. Process in turn each edge that has not previously been traversed as follows: an edge incident to the root becomes a flatstep, otherwise a leftmost edge (from its parent vertex) becomes an upstep, a rightmost edge becomes a downstep, and an interior edge becomes a flatstep. Finally, delete the initial step (necessarily a flatstep). For example, the bush in Figure 5a (with edges numbered in the order they are first encountered) corresponds to the Motzkin path AB in Figure 5b. Also, the 4 bushes in Figure 4 correspond respectively to the 4 Motzkin paths in Figure 1.

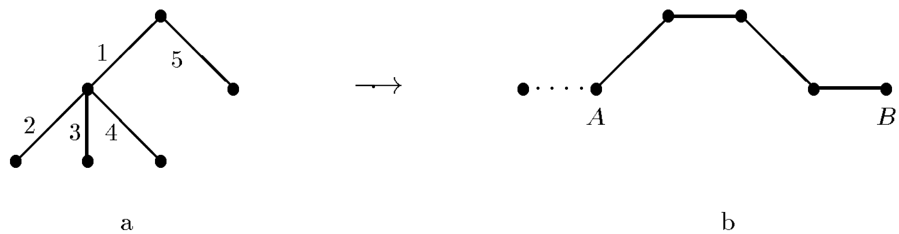


Figure 5

The inverse mapping is somewhat trickier to describe. A little terminology for Motzkin paths is helpful. We distinguish between *ground-level* and *raised* flatsteps. Each downstep has an *associated* upstep: head straight west from the downstep until you encounter an upstep. Similarly, each raised flatstep has an associated upstep: start just beneath the flatstep and head west. Thus in Figure 6, the downsteps 5 and 7 are associated respectively to upsteps 4 and 2 while the raised flatsteps 3 and 6 are both associated to upstep 2. (Nothing is associated to the ground-level flatsteps 1 and 8.)

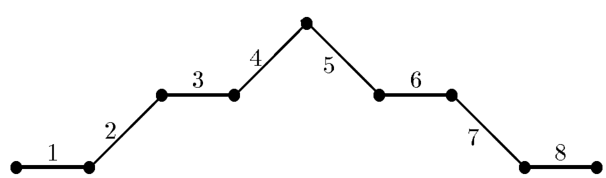


Figure 6

Now given a Motzkin path, prepend a (ground-level) flatstep and turn each step of the path into an edge of a bush, working from left to right, as follows. Each new edge will be joined to an existing vertex (parent) thus introducing a new vertex (child) which will always be placed to the right of any existing children of the parent. First create the root. Proceeding, a ground-level flatstep becomes an edge from the root. An upstep becomes an edge from the vertex introduced by the previous step (and this is the only way a *first* child of a non-root parent arises). Finally, a raised flatstep or a downstep introduces a (right) sibling to the vertex introduced by its associated upstep. Since upsteps and downsteps come in associated pairs, this ensures that each non-root parent vertex will have at least 2 children. It is not hard to see that these mappings are indeed inverses of one another.

3. Schröder numbers and royal paths

Counting bushes by number of leaf vertices (rather than by number of edges) yields the Schröder numbers [5]: S_n is the number of bushes with $n+1$ leafs. Under the bijection of the preceding section, a leaf vertex in a bush corresponds to a path vertex from which an upstep does *not* emanate. Noting that the terminal point of a path is always such a vertex, we conclude that S_n is the number of Motzkin paths with a total of n flatsteps and downsteps. Flipping Motzkin paths across ground level and rotating 45° counterclockwise, we get the more common interpretation of S_n as the number of lattice paths from $(0,0)$ to (n,n) consisting of steps $(1,0),(0,1),(1,1)$ that never rise above the line $y=x$ (called *royal* paths in [6]). The $S_2=6$ bushes with 3 leaf vertices are given below their corresponding royal paths in Figure 7.

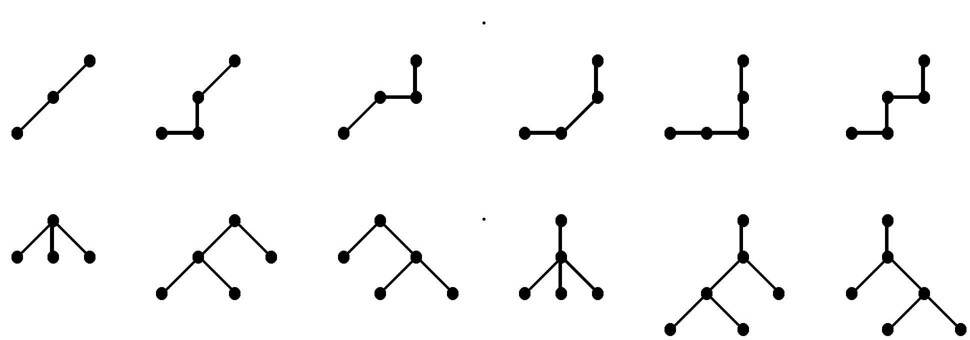


Figure 7

The Schröder number sequence $(S_n)_{n\geq 0}$ begins $1, 2, 6, 22, 90, 394, \ldots$ and S_n is even for $n \geq 1$. The "bush" interpretation of Schröder numbers makes this obvious: there are as many planted (i.e. root has degree one) bushes on n leaf vertices as non-planted ones (simply remove the root edge from the planted bushes). This observation has the following translation to royal paths: the S_n royal paths from $(0,0)$ to (n,n) split into 2 equal-size classes according as they possess a diagonal step on the line $y=x$ or not. For example, the first 3 royal paths in Figure 7 do so and the second 3 don't. A concluding exercise: find a bijection between these equal-size classes. (Hint. Look for the *last* diagonal step on the line $y=x$, and failing that, look for the *first* return to the line $y=x$.)

References

1. M. Aigner, *Motzkin numbers*, Europ. J. Combinatorics **19** (1998), 663–675.
2. D. Callan, *Pair them up!: A visual approach to the Chung-Feller theorem*, Coll. Math. J. **26** (1995), 196–198.
3. R. Donaghey and L. W. Shapiro, *Motzkin numbers*, J. Comb. Th. A **23** (1977), 291–301.
4. R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1994.
5. D.G. Rogers and L. W. Shapiro, *Combinatorial Mathematics, Lecture Notes in Mathematics*, vol. 686, Springer-Verlag, New York, 1978, pp. 267–274.
6. Sloane and Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, New York, 1995.