Permutations Avoiding a Nonconsecutive Instance of a 2- or 3-Letter Pattern

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Abstract

We count permutations avoiding a nonconsecutive instance of a two- or three-letter pattern, that is, the pattern may occur but only as consecutive entries in the permutation. Two-letter patterns give rise to the Fibonacci numbers. The counting sequences for the two representative three-letter patterns, 321 and 132, have respective generating functions $(1+x^2)(C(x)-1)/(1+x+x^2-xC(x))$ and $C(x+x^3)$ where $C(x)$ is the generating function for the Catalan numbers.

1. Introduction

There is a large literature on pattern avoidance in permutations and words. Problems treated include counting permutations avoiding a pattern or set of patterns, or containing patterns a specified number of times. Or a pattern may be allowed but only as part of a larger pattern. Pattern occurrences may be unrestricted (the classical case), or required to be consecutive (i.e., contiguous entries in the permutation), or a mixture of the two. Here we consider the following variation. Given a pattern π, how many permutations on $[n]$ avoid nonconsecutive instances of π, that is, the pattern π may occur but only as consecutive entries in the permutation. In other words, *all* occurrences of π are required to be of the consecutive type. As usual, for $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ a permutation on $[k]$, an instance of π in a permutation $(a_1, a_2, \ldots, a_n)$ is a subpermutation $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$, $1 \leq i_1 < \ldots < i_k \leq n$, whose reduced form (replace smallest entry by 1, next smallest by 2, and so on) is π. In the present paper we deal with patterns of length ≤ 3. The operations reverse and complement

$(a_i \to n+1-a_i)$ on permutations and patterns preserve this type of pattern avoidance; so we need only consider representative patterns in the orbits under the action of the group they generate. For patterns of length ≤ 3, this reduces the problem (as usual) to the patterns 21, 321 and 132. The following sections treat them one at a time.

2. The 21 pattern

Avoiding a nonconsecutive 21 means that every inversion arises from a contiguous pair of entries. Such permutations on $[n]$ are all obtained from the identity permutation $(1, 2, \ldots, n)$ by selecting a set of non-overlapping pairs of consecutive integers in $[n]$ and, for each pair, switching the entries in those 2 positions. The first entries of these pairs form a scattered subset of $[n-1]$, that is, a subset for which distinct elements differ by at least 2. It is well known that such subsets are counted by the Fibonacci number F_{n+1}. Hence we have

Theorem 1. *The number of permutations on* $[n]$ *that avoid a nonconsecutive* 21 *pattern is* F_{n+1}.

3. The 321 pattern

Let us introduce some notation:

- $\mathcal{A}_n$ is the full set of permutations on $[n]$ that avoid a nonconsecutive 321.
- $\mathcal{B}_n, \mathcal{D}_n$ are, respectively, the permutations in $\mathcal{A}_n$ that start with a 321 in the first 3 positions, and those that don't. Thus $|\mathcal{A}_n| = |\mathcal{B}_n| + |\mathcal{D}_n|$.
- $\mathcal{C}_n$ is the set of permutations in $\mathcal{A}_n$ that contain no 321 at all, the set of so-called 321-avoiding permutations. It is well known [1] that $|\mathcal{C}_n| = C_n$, the nth Catalan number.
- $\mathcal{A}_{n,k}$, $(1 \leq k \leq n-2)$ is the set of permutations in $\mathcal{A}_n$ that contain one or more (necessarily consecutive) 321s, the first of which starts at position k. Thus $|\mathcal{A}_n| = C_n + \sum_{k=1}^{n-2} |\mathcal{A}_{n,k}|$.

First, observe that a permutation in $\mathcal{B}_n$ $(n \geq 3)$ must have 2 as its second entry and 1 as its third entry (else an offending pattern would be present). Deleting these entries and subtracting 2 from all other entries is a bijection to $\mathcal{D}_{n-2}$. Hence $|\mathcal{B}_n| = |\mathcal{D}_{n-2}|$. Next, given $\rho \in \mathcal{A}_{n,k}$ $(2 \le k \le n-2)$, we can form two new permutations σ, τ as follows. Take the first $k-1$ entries and the entry in position $k+2$ (necessarily comprising the first k positive integers) to form σ. Delete the first $k-1$ entries and reduce (replace smallest entry by 1, next smallest by 2, and so on) to form τ. We leave the reader to verify that this is a bijection $\mathcal{A}_{n,k} \to \mathcal{C}_k \times \mathcal{B}_{n-k+1}$. Hence $|\mathcal{A}_{n,k}| = C_k|\mathcal{B}_{n-k+1}| = C_k|\mathcal{D}_{n-k-1}|$.

Now, with a lowercase letter a_n denoting the size of $\mathcal{A}_n$ and so on, we have just shown that $a_n = C_n + \sum_{k=1}^{n-2} C_k d_{n-k-1}$ (since $C_1 = 1$). But also $a_n = b_n + d_n$ and $b_n = d_{n-2}$. Eliminating a_n and b_n and reindexing yields

$$d_n = C_n + \sum_{k=1}^{n-3} C_{k+1} d_{n-2-k},$$

a recurrence for d_n that is valid for $n \ge 1$ and hence determines d_n (no initial condition necessary). Since the sum is a convolution, this recurrence routinely yields the generating function

$$D(x) = \frac{C^*(x)}{1 + x^2 - xC^*(x)},$$

where $D(x) := \sum_{n\ge1} d_n x^n$ and $C^*(x) := \sum_{n\ge1} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} - 1$ (the superscript * indicating that C_0 is omitted from the sum). The formula for a_n then yields

Theorem 2. *The number a_n of permutations on $[n]$ that avoid a nonconsecutive* 321 *pattern has generating function*

$$\sum_{n\ge1} a_n x^n = \frac{C^*(x)}{1 - \frac{x}{1+x^2} C^*(x)}.$$

The sequence $(a_n)_{n\ge1}$ begins 1, 2, 6, 18, 56, 182, 607, 2064, ...

3. The 132 pattern

First, observe that 132-avoiding permutations are characterized by the property that, for each entry a, the set of succeeding entries that are $< a$ form an initial segment of the positive integers. Also, Simion and Schmidt [2] gave a bijection (see [3] for an equivalent description) from 132-avoiding to 123-avoiding permutations, the latter corresponding to 321-avoiding under reversal, and so 132-avoiding permutations are also counted by the Catalan numbers.

Now let $\mathcal{E}_n$ denote the set of permutations on $[n]$ that avoid a nonconsecutive 132 and $\mathcal{E}_{n,k}$ the permutations in $\mathcal{E}_n$ with k (necessarily consecutive) 132s. Consider a permutation in $\mathcal{E}_{n,k}$. First, record the positions $i_1, i_2, \ldots, i_k$ of the middle entries of its 132 patterns—they form a subset of the interval $[2, n-1]$ whose entries all differ by at least 3 because, as is easily checked, the 132 patterns cannot overlap. Then delete the first and last entry of each 132 pattern and reduce. The result is a 132-avoiding permutation on $[n-2k]$. Thus each permutation in $\mathcal{E}_{n,k}$ yields a set $\{i_1, i_2, \ldots, i_k\} \subset [2, n-1]$ with entries differing by at least 3 together with a 132-avoiding permutation on $[n-2k]$.

Conversely, given any such set and permutation, we can construct a permutation in $\mathcal{E}_{n,k}$ that yields them. Take, for example, $n = 10$ and $k = 2$ and suppose given $\{i_1, i_2, \ldots, i_k\} = \{4, 8\}$ and 132-avoiding permutation $(6, 5, 3, 4, 2, 1)$. Place blanks in the positions immediately neighboring each i_j and place the permutation entries in the remaining positions:

1	2	3	4	5	6	7	8	9	10
6	5		3		4		2		1

Leave intact the entries after the entry in position i_1 and smaller than it (as noted above, these entries form an initial segment $1, 2, \ldots, j$ of the positive integers), place $j+1$ in position $i_1 - 1$, $j+2$ in position $i_1 + 1$, and increase all other entries by 2:

1	2	3	4	5	6	7	8	9	10
8	7	3	5	4	6		2		1

Proceed likewise left to right to fill in the remaining pairs of blanks. It is easy to check that the result, here $(10, 9, 5, 7, 6, 8, 2, 4, 3, 1)$, is a permutation in $\mathcal{E}_{n,k}$ that yields the given $\{i_1, i_2, \ldots, i_k\}$ and 132-avoiding permutation under the process of the previous paragraph.

There are $\binom{n-2k}{k}$ such sets $\{i_1, i_2, \ldots, i_k\}$ and C_{n-2k} 132-avoiding permutations on $[n-2k]$ and so $|\mathcal{E}_{n,k}| = \binom{n-2k}{k} C_{n-2k}$. Summing over k and taking into account the permutations that avoid 132 altogether, we have

Theorem 3. *The number of permutations on $[n]$ that avoid a nonconsecutive 132 pattern is*

$$\sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-2k}{k} C_{n-2k}.$$

Routine manipulations lead to a succinct generating function:

$$\begin{aligned}\sum_{n\geq 0}|\mathcal{E}_n|x^n &= \sum_{n,k\geq 0}\binom{n-2k}{k}C_{n-2k}x^n \\ &= \sum_{k\geq 0}\left(\sum_{n\geq 3k}\binom{n-2k}{k}C_{n-2k}x^{n-3k}\right)x^{3k} \\ &= \sum_{k\geq 0}\left(\sum_{n\geq k}\binom{n}{k}C_n x^{n-k}\right)x^{3k} \\ &= \sum_{k\geq 0}\frac{C^{\{k\}}(x)}{k!}x^{3k} \\ &= C(x+x^3),\end{aligned}$$

the last equality by Taylor's theorem, where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. So we have

Corollary. *The generating function for permutations that avoid a nonconsecutive* 132 *pattern is*

$$\frac{1-\sqrt{1-4x-4x^3}}{2(x+x^3)}.$$

The counting sequence ($n\geq 0$) begins 1, 1, 2, 6, 18, 57, 190, 654, 2306, ...

Added in Proof These results have previously been obtained in a wider context by Anders Claesson [4].

References

[1] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/ .

[2] Rodica Simion and Frank W. Schmidt, Restricted Permutations, *Europ. J. Combinatorics* **6** (1985), 383-406.

[3] David Callan, A Wilf Equivalence Related to Two Stack Sortable Permutations, 2005, preprint, http://front.math.ucdavis.edu/math.CO/0510211 .

[4] Anders Claesson, Counting segmented permutations using bicoloured Dyck paths, *Elec. J. Comb.* ,Vol. 12, **R39**, 2005.