Bijections for the Identity $4^n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k}$

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The generating function for the even central binomial coefficients $\{\binom{2k}{k}\}_{k\geq 0}$ is $(1-4x)^{-1/2}$ and so the title identity follows by equating coefficients in the power series identity $(1-4x)^{-1} = (1-4x)^{-1/2}(1-4x)^{-1/2}$. Donald Knuth featured the title identity in his introduction to the book **A = B** [1]: "Eventually I learned a tricky way to prove (it), but if I had known the methods in this book I could have proved the identity immediately". Still, a simple binomial coefficient identity should have a simple combinatorial interpretation and this one has (at least) two, both mentioned in Feller [2]: it counts the 2^{2n} $2n$-step lattice paths of upsteps (1,1) and downsteps $(1,-1)$ (a) by number $2k$ of steps before the path's last return to ground level, and (b) by number $2k$ of steps lying above ground level. Combinatorial proofs of (a) are well known [3, 4]. After reviewing several of them, we give a bijection to prove (b).

First, a little terminology: ground level is the horizontal line through a path's initial point (the x-axis if the path starts at the origin). A *first quadrant* path is one that never dips below ground level and similarly, a *fourth quadrant* path is one that never rises above ground level. A *balanced* path is one that ends at ground level, and so has equal numbers of upsteps and downsteps. A *no-return* path is one that never returns to ground level after the initial point.

There is a simple bijection from even-length no-return paths to even-length first quadrant paths: it's the identity on no-return paths that start with an upstep—they are already first quadrant paths. Given a no-return path that starts with a downstep, first flip the entire path in ground level, then flip the rightmost upstep at height level 2 as illustrated in Figure 1.

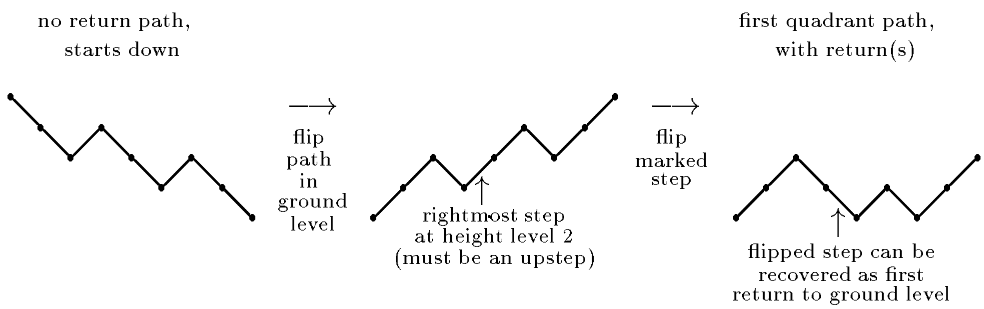


Figure 1

The result is a first quadrant path with at least one return to ground level.

The next bijection, from even-length first quadrant paths to balanced paths, is due to Nelson [2, p. 67, Problem 7], also given in [5], and illustrated in Figure 2.

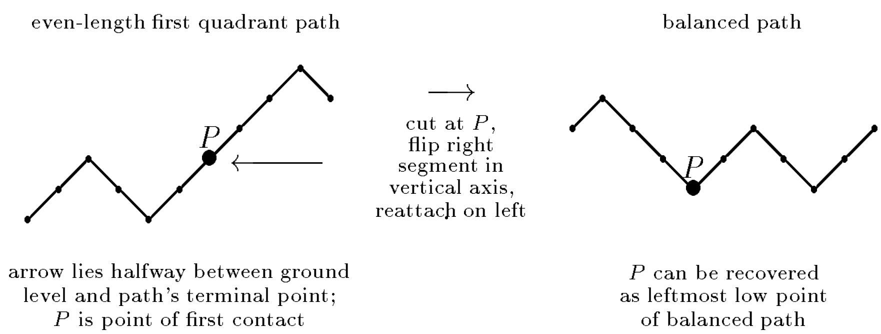
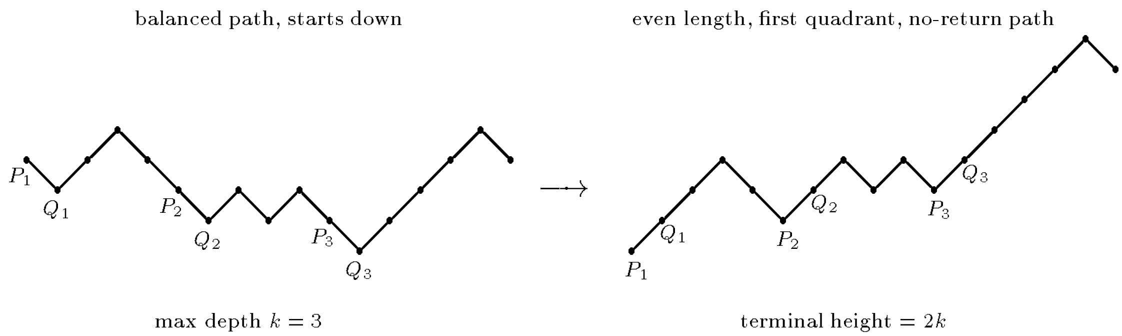


Figure 2

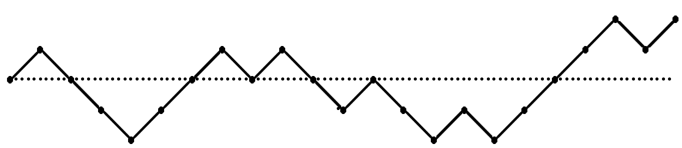
The composition of these two bijections gives a bijection from $2n$-step no-return paths to $2n$-step balanced paths and interpretation (a) follows.

Here is a direct bijection between these sets, modelled after the one in Kleitman [3]. It sends balanced paths B that start with a downstep to first quadrant no-return paths. Let k be the depth of the lowest point on B. For each j, $1 \leq j \leq k$, locate the leftmost point Q_j at depth j and the point P_j immediately preceding Q_j, and flip the downstep P_jQ_j to an upstep.



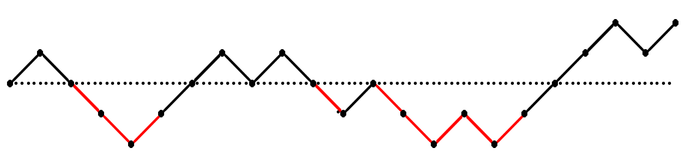
To invert the map, recover P_j as the rightmost point at height $j-1$, $1 \le j \le h/2$, where h is the terminal height of an even length first quadrant no-return path.

The main bijection, to establish interpretation (b), is from $2n$-step paths with $2k$ steps above ground level to pairs of paths, the first a $2k$-step first quadrant path, the second a $2(n-k)$-step fourth quadrant path. The first-quadrant-to-balanced bijection converts both restricted-quadrant paths to balanced paths (of course, the fourth quadrant path is first flipped across ground level) and (b) follows. So suppose given a $2n$-step path P with $2k$ steps above ground level.

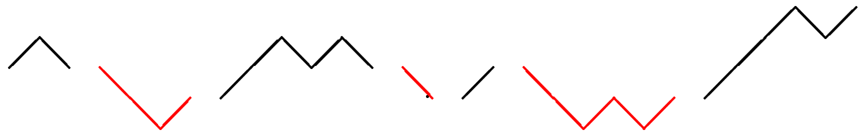


P

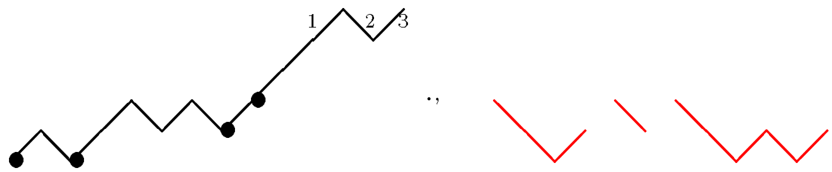
As a first stab, we might simply strip off the segments of P lying above ground level and concatenate. This would certainly produce paths of the right length in the right quadrants. Clearly, though, it does not preserve enough information to reconstruct the original path. But with a little tweaking, this strategy turns out to be reversible. First, color all steps below ground level red *except* those upsteps that return the path to ground level.



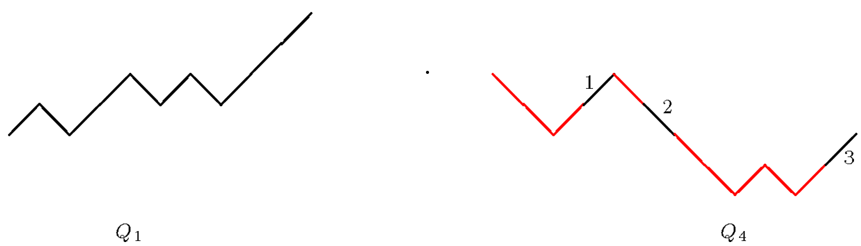
Next, separate the red and black segments of the path, maintaining left to right order.



Now, close up the black segments to form a first quadrant path. The steps it has picked up from below ground level (= # red segments, here 3) will be "paid back" from the end; so the last 3 steps are numbered.



Remove the numbered steps from the first quadrant path and distribute them, one apiece and in order, at the right end of the red segments, and close up. The results (ignoring color and numbering) are the desired first and fourth quadrant paths Q_1 and Q_4.



The invertibility of this map is not obvious and presenting its inverse takes a little work. A *Dyck* path is a balanced first quadrant path. A *strict* Dyck path is an upstep followed by a Dyck path followed by a downstep. Let us say a *modified* Dyck path is an upstep followed by a Dyck path followed by either an upstep or a downstep.

Lemma 1 *Every nonempty even length first quadrant path P decomposes uniquely as a concatenation of modified Dyck paths.*

Proof. If an initial segment of P is a Dyck path, the shortest such (a strict Dyck path) is the first modified Dyck path in the decomposition. Otherwise, the last point on P at height 1 must be followed by an upstep S (since P, being balanced, terminates at even height) and the initial segment of P through step S is the first path in the decomposition. In either case, the segment of P following this modified Dyck path (if any) is again an even length first quadrant path, and existence follows by induction. Uniqueness is clear.

We note incidentally that Lemma 1 gives another bijection from even length first quadrant to balanced paths: in the modified Dyck path decomposition, leave the strict Dyck paths intact and flip the others (except for their last steps) to turn them into inverted strict Dyck paths. In the resulting balanced path, the division points of the modified Dyck path decomposition can be recovered as the returns to ground level.

Of course, an even length fourth quadrant path Q_4 has an analogous decomposition into "inverted" modified Dyck paths whose last steps we call the *critical* steps of Q_4.

Now we can define the inverse of the above map $P \rightarrow (Q_1, Q_4)$. Given a pair (Q_1, Q_4) of even length first/fourth quadrant paths, build up P as follows.

- **Initialize** Remove the longest initial segment D_0 (possibly empty) of Q_1 that is a Dyck path and set $P = D_0$. Now Q_1 is a no-return path.
- **Main process** While Q_1 and Q_4 are both nonempty, remove the initial segment Aa of Q_4 through its first critical step a. Append a to Q_1 (making Q_1 an odd length path) and then remove from Q_1 its longest initial segment B that ends at height 1. Append A, B (in that order) to P. Now Q_1 and Q_4 are once again of even length and P is still balanced. (Repeat till one of Q_1,Q_4 is empty.)
- **Finish** When one of Q_1,Q_4 becomes empty, append the other (possibly also empty) to P and stop.

References

[1] Marko Petkovsek, Herbert S. Wilf, Doron Zeilberger, $\boldsymbol{A = B}$, A K Peters, Wellesley, Mass., 1996.

[2] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd ed.*, John Wiley and Sons, New York, 1968.

[3] Daniel J. Kleitman, A Note on some subset identities, *Studies in Applied Mathematics* **LIV**, 1975, 289–292.

[4] Marta Sved, Counting and recounting: the aftermath, *Math. Intelligencer* **6**, No. 4, 1984, 44–45.

[5] Norbert Hungerbühler, Bijection between certain lattice paths, *Math. Magazine* **71**, 1998, 219.