Priority queues and the Bruhat order

DAVID CALLAN

Department of Statistics
University of Wisconsin-Madison
Medical Science Center
1300 University Ave
Madison, WI 53706-1532

callan@stat.wisc.edu

July 5, 2012

Abstract

Valid input-output pairs for priority queue sorting—the "Insert / DeleteMaximum" paradigm—are characterized in terms of paths in the Hasse diagram for the weak Bruhat order on permutations.

The priority queue process on a permutation works as follows. Start with a permutation (the initial input) on the left, a container in the middle and an output permutation, initially empty, on the right. A step consists of transferring the last entry of the current input to the container or, providing the container is not empty, transferring the largest element in the container to the start of the current output. The process terminates when all entries are transferred to the output permutation.

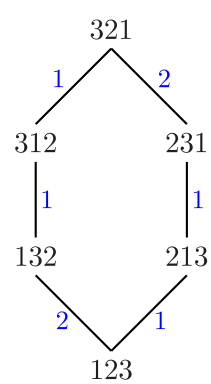
a priority queue process

Every permutation can be sorted into increasing order by first feeding all its entries into the container and then removing them one by one, but a permutation can be sorted into decreasing order only if it is decreasing to begin with. A pair of permutations (p, q) is said to be *allowable* if q can be obtained from p by a priority queue process.

The number of allowable pairs of permutations on $[n] := [1, n] = \{1, 2, \ldots, n\}$ is $(n + 1)^{n-1}$, the number of labeled trees on $[0, n]$, and several bijective proofs are known

[1, 3, 4, 5]. There is a characterization of allowable pairs in terms of simultaneous pattern avoidance as follows [2]. A *pattern* is a permutation on an initial segment of the positive integers. For a permutation p on $[n]$, each nonempty subset A of $[n]$ gives rise to an instance of a pattern in p: arrange A in the order in which its entries occur in the permutation, and then replace the smallest entry by 1, next smallest by 2 and so on. For example, the subset $\{2, 3, 5\}$ of [5] occurs in the permutation 43215 in the order 325, thereby giving an instance of the pattern 213. A pair of permutations (p, q) on $[n]$ is said to (simultaneously) avoid the pair of patterns (σ, τ) if no subset of $[n]$ gives rise to both an instance of σ in p and of τ in q. For example, $\{2, 3, 5\}$ forms a 321 pattern in 15432 and at the same time a 213 pattern in 43215, and so the pair (15432, 43215) is not $(321, 213)$-avoiding, while (4132, 2134) *is* $(321, 213)$-avoiding even though neither permutation avoids its corresponding pattern. (The point is that you can get a subset of [4] that gives a 321 pattern in the first permutation and another subset that gives a 213 pattern in the second but you can't get a single subset to do both simultaneously.) In these terms, a pair of permutations (p, q) is allowable if and only if it avoids both the pairs of patterns (12, 21) and (321, 213).

In this note, we give another characterization of allowable pairs in terms of the weak Bruhat order on S_n, the set of permutations on $[n]$: $p \prec q$ in this order if q can be obtained from p by a sequence of one or more interchanges of two adjacent entries, each of which increases the number of inversions by 1. Let $\mathcal{H}_n$ denote the edge-labeled Hasse diagram for the weak Bruhat order on S_n, where each edge is labeled by the smaller of the two interchanged entries associated with the edge; $\mathcal{H}_3$ is shown.



edge-labeled Hasse diagram for the weak Bruhat order on S_3

We can now characterize allowable pairs in S_n as follows.

Proposition 1. *A pair of permutations* (p, q) *in* S_n *is valid input-output for a priority queue process as above if and only if* (i) $q \preceq p$ *in the weak Bruhat order and* (ii) *there exists a path in* $\mathcal{H}_n$ *going up from* q *to* p *that has weakly decreasing labels.*

For example, the only path from $q = 213$ up to $p = 321$ has labels 1,2 (in that order), a sequence that fails to be weakly decreasing, and indeed $(p, q) = (321, 213)$ is not an allowable pair.

If $q \preceq p$ in $\mathcal{H}_n$ and there is a weakly decreasing path from q up to p, then it is unique. So there is a manifestation of $(n + 1)^{n-1}$ in the weak Bruhat order:

Proposition 2. *Turn* $\mathcal{H}_n$ *into an edge-labeled digraph by directing all edges upward. Then* $(n + 1)^{n-1}$ *counts weakly decreasing paths in* $\mathcal{H}_n$.

The priority queue paradigm is, in short, “insert-deleteMax”. Sometimes deleteMax is replaced by deleteMin. In this case, the characterization of allowable pairs in Prop. 1 still holds provided edges are labeled by the *larger* of the two associated interchanged entries, and the pattern avoidance criterion becomes “avoid (21, 12) and (123, 231)”. Other variants include inserting the *first* entry of the input permutation into the container or placing the deleted entry at the *end* of the output permutation (or both) giving further variants on the two pairs of patterns to be avoided.

References

[1] M.D. Atkinson and R. Beals, Priority queues and permutations, *SIAM J. Comput.* **23** (1994) 1225–1230. MR1303332 (95i:68094)

[2] M.D. Atkinson, S.A. Linton, and L.A. Walker, Priority queues and multisets, *Elect. J. Combin.* **2** (1995) #R24. MR1358957 (96j:68039)

[3] M. Golin and S. Zaks, Labelled trees and pairs of input-output permutations in priority queues, *Theoret. Comput. Sci.* **205** (1998) 99–114. MR1638640 (99f:68036) MR1897927 (2003c:05013)

[4] A. M. Hamel, Priority queue sorting and labeled trees, *Ann. Comb.* **7** (2003), no. 1, 49–54. MR1984645 (2004c:05010)

[5] L. Kalikow, Enumeration of parking functions, allowable permutation pairs, and labeled trees, Ph.D. Thesis, Brandeis University, 1999.