Riordan Numbers Are Differences of Trinomial Coefficients

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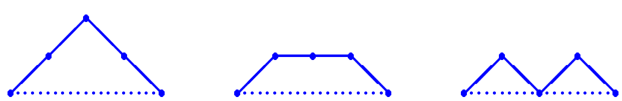
The Riordan number R_n [A005043] is the number of Riordan paths of length n where a Riordan path is a Motzkin path [A001006] containing no flatsteps at ground level. Thus $R_4 = 3$ counts



The trinomial coefficient $t(n,k)$, $0 \le k \le n$ is the coefficient of x^k in $(x^{-1}+1+x)^n$, and $\big(t(n,0)\big)_{n\ge 0} = (1,1,3,7,19,51,\dots)$ is the central trinomial coefficient [A002426]. Clearly, $t(n,k)$ counts the set $\mathcal{T}(n,k)$ of lattice paths consisting of upsteps $U=(1,1)$, flatsteps $F=(1,0)$ and downsteps $D=(1,-1)$ such that the path (i) contains n steps, and (ii) ends k units above ground level, the horizontal line through its initial point. Thus $\mathcal{T}(n,0)$ is the set of n-step *balanced* UFD-paths and a Motzkin path is a *nonnegative* balanced UFD-path.

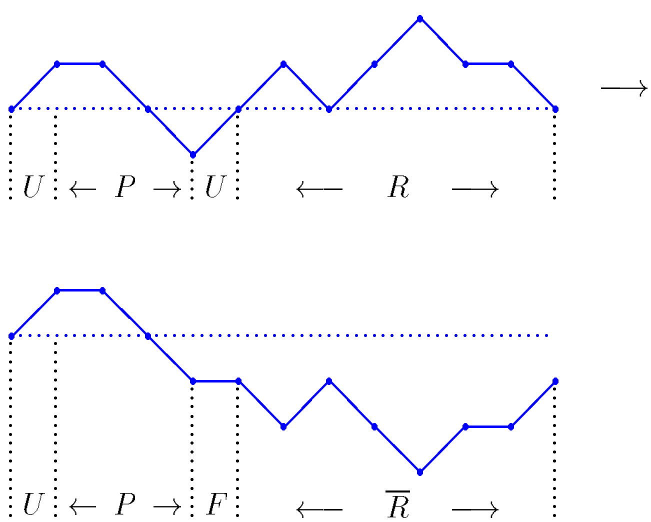
Just as the Catalan number is the difference of a central binomial coefficient and its predecessor, the Riordan number is the difference of a central trinomial coefficient and its predecessor:

Theorem. $R_n = t(n,0) - t(n,-1)$.

Since the set $\mathcal{R}_n$ of paths counted by R_n is a subset of $\mathcal{T}(n,0)$, it suffices to give a bijection from $\mathcal{T}(n,0)\backslash\mathcal{R}_n$ to $\mathcal{T}(n,-1)$, that is, to the n-step UFD-paths that end 1 unit below ground level. So suppose given a balanced UFD-path of length n that either has at least one F at ground level or dips below ground level at some point (or both). If the path ends with a U or F, rotate this step 45°, that is, $U \to F$, $F \to D$. This produces, in reversible fashion, all paths in $\mathcal{T}(n,-1)$ that end F or D. If the given path ends with a D, however, consider its first step and, if it is U, let R denote the longest Riordan subpath that terminates the given path. Note that R is necessarily nonempty and not the entire path. Also, the step immediately preceding R in the given path cannot be D (this would violate the maximality of R). Thus, in case it ends D, the given path has one of the following four mutually exclusive forms (P a UFD-subpath), and they are mapped as indicated:

$$\begin{array}{rcl} DPD & \to & F\overline{P}U \\ FPD & \to & D\overline{P}U \\ UPFR & \to & UPD\overline{R} \\ UPUR & \to & UPF\overline{R} \end{array}$$

Here, $\overline{P}$ is the result of flipping P across a horizontal line. An example for the last form is illustrated.



We leave to the reader the easy verification that this map is invertible and therefore is the desired bijection.