$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} \left\{ {k+1 \atop j} \right\} \binom{n}{j} (j-1)!$$

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Consider the set of sequences of $k+1$ positive integers, all $\le n$, such that the largest entry occurs in the last position (and possibly elsewhere).

On the one hand, $\sum_{i=1}^{n} i^k$ counts these sequences by *last entry* i because there are i choices for each of the k preceding positions.

On the other hand, $\sum_{j=1}^{k+1} \left\{ {k+1 \atop j} \right\} \binom{n}{j} (j-1)!$ counts them by *number* j *of distinct integers* occurring in the sequence because each such sequence can be formed (uniquely) as follows. Choose a j-subset of $[n]$ to serve as the entries appearing in the sequence—$\binom{n}{j}$ choices. Choose a permutation of this j-set such that its largest entry occurs last—$(j-1)!$ choices—to serve as the permutation obtained from the sequence by erasing all but the last occurrence of each integer appearing in the sequence. Choose a partition of the positions $1, 2, \ldots, k+1$ into j nonempty blocks—$\left\{ {k+1 \atop j} \right\}$ choices—and order the blocks in increasing order of largest entry. These choices serve to specify the sequence if we place the ith entry of the permutation into every position in the ith block. For example with $n = 7, k = 8, j = 4$ the j-set $\{2, 4, 6, 7\}$, permutation $6\,2\,4\,7$, and partition 15-6-37-2489 leads to the sequence (6,7,4,7,6,2,4,7,7), and the process is reversible.