Unimodality of Trinomial Coefficients

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A recent Note [1] turns on the mound-shaped (symmetric unimodal) property of the coefficients of $(x^{-1}+1+x)^n = \sum_{j=-n}^{n} N_j^{(n)} x^j$. Unimodality for $n \geq 2$ asserts $N_j^{(n)} < N_{j-1}^{(n)}$, $1 \leq j \leq n$, and a proof by induction is suggested in [1]. Here is a combinatorial proof. Clearly, $N_j^{(n)}$ counts the set $\mathcal{N}_j^{(n)}$ of n-step paths consisting of upsteps (corresponding to exponents $+1$), flatsteps (0) and downsteps (-1), and containing j more upsteps than downsteps. To show $N_j^{(n)} \leq N_{j-1}^{(n)}$, we exhibit an injection $\mathcal{N}_j^{(n)} \to \mathcal{N}_{j-1}^{(n)}$. Given a path $P \in \mathcal{N}_j^{(n)}$, let $k \in [0,j]$ denote the number of terminal downsteps in P. Flip these downsteps to upsteps and rotate the immediately preceding step (an upstep or flatstep) 45° clockwise. Then pick out the leftmost of the remaining original upsteps on k successive levels starting at the highest level and flip these k upsteps to downsteps. The net effect is to lower the path's terminal point by one unit, placing it in $\mathcal{N}_{j-1}^{(n)}$. Two examples are illustrated in Figure 1 (blue and red giving two alternatives ;both paths with $k=2$, heavy line for the rotated step, and numerals on the flipped upsteps).

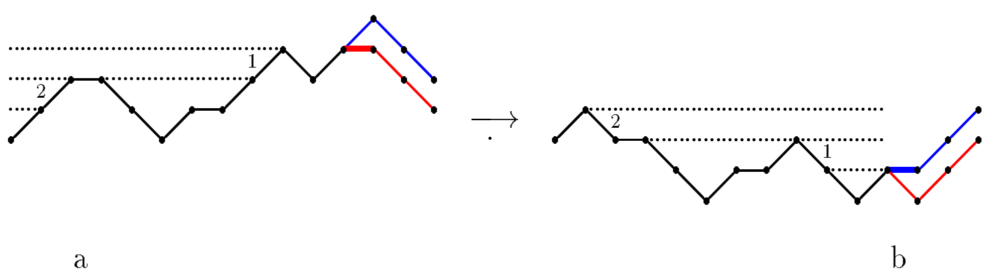

Figure 1

To reverse the mapping (thus establishing injectivity), the flipped upsteps can be retrieved in the rightmost positions on the k highest levels (ignoring the new terminal upsteps) as in Figure 1b.

References

[1] J. Marshall Ash, The probability of a tie in an n-game match, *Amer. Math. Monthly* **105**, 1998, 844–846.