Why is $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n}\binom{2n}{n-1}$?

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
1210 W. Dayton St
Madison, WI 53706-1693
callan@stat.wisc.edu

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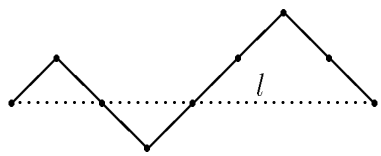
1. Introduction The Catalan number C_n is the cardinality (number of elements) of the set $\mathcal{D}_n$ of Dyck paths or "mountain ranges" of size n—the lattice paths of n upsteps and n downsteps that never dip below "ground level". Writing an upstep as $+1$ and a downstep as -1, a Dyck path can be represented as a sequence of n 1's and n -1's whose partial sums are all nonnegative. We will review three methods of counting Dyck paths of size n leading, respectively, to the three expressions in the title (each of which is equal to C_n).

Each of the title expressions consists of a binomial coefficient b divided by a divisor d. These divisors are $n+1$, $2n+1$ and n respectively. In each case there is a class $\mathcal{L}$ of lattice paths whose cardinality is the binomial coefficient b, and a way of coding these paths as sequences of length d with a special property: a code sequence cyclically rotated is again a valid code sequence. In each case, there is a parameter ν defined on $\mathcal{L}$ (equivalently, on its code sequences) with the following property: ν has d possible values and on the d sequences obtained by cyclically rotating a given code sequence, ν takes on each of its possible values exactly once. Thus $\mathcal{L}$ is partitioned into $\frac{b}{d}$ cyclic-rotation equivalence classes each of size d. The sequences for which ν has its maximum possible value are the Dyck paths of size n (first case) or are in a simple one-to-one correspondence with them (the other two cases). Thus there is one Dyck path per equivalence class and so $|\mathcal{D}_n| = b/d$, the quotient of the binomial coefficient and its divisor.

2. Case $\mathbf{d = n+1}$ Here $\mathcal{L}(=\mathcal{L}_1)$ is the set of all $\binom{2n}{n}$ lattice paths of n upsteps and n downsteps. For such a path, the n downsteps serve to separate $n+1$ *ascents* each consisting of (zero or more) contiguous upsteps. The path is coded by the lengths of these ascents (see Figure 1).

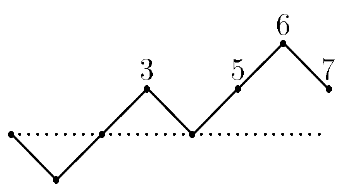


$n = 4$, ascent length sequence=(1,0,3,0,0), $\nu = 3$ upsteps above ground level

Figure 1

The parameter ν is the number of upsteps above the x-axis and ν ranges over $[0, n]$. The paths where ν has its maximal value n actually *are* the Dyck paths of size n.

3. Case $\mathbf{d = 2n + 1}$ Here $\mathcal{L}(= \mathcal{L}_2)$ is the set of all $\binom{2n+1}{n}$ lattice paths of $n + 1$ upsteps and n downsteps. Here a path is coded simply by its sequence of ± 1's. The parameter ν is the number of the path vertices strictly above the x-axis (or the number of positive partial sums of its ± 1 sequence, see Figure 2).



± 1 sequence $= (-1, 1, 1, -1, 1, 1, -1)$
partial sums $= (-1, 0, 1, 0, 1, 1, 1)$ with positive entries at positions 3,5,6,7
$\nu = 4$ vertices above ground level

Figure 2

The terminal point is always above the x-axis, so ν ranges over $[1, 2n+1]$. The paths with $\nu = 2n+1$ correspond bijectively to Dyck paths of size n *via*: delete first step (necessarily an upstep).

4. Case $\mathbf{d = n}$ Here $\mathcal{L}(= \mathcal{L}_3)$ is the set of all $\binom{2n}{n-1}$ lattice paths of $n + 1$ upsteps and $n - 1$ downsteps. Here a path is coded by by the lengths of its ascents as in the first case except now, with only $n - 1$ downsteps in a path, the code squences are of length n The parameter ν is the number of upsteps above the line $y = 1$ and this time ν ranges over $[1, n]$. The paths with $\nu = n$ correspond bijectively to Dyck paths of size n *via*: flip the rightmost upstep between levels $y = 1$ and $y = 2$ (there must be one!) to a downstep.

To reverse this map, locate the last downstep that returns a Dyck path to the x-axis and flip it to an upstep (see Figure 3).

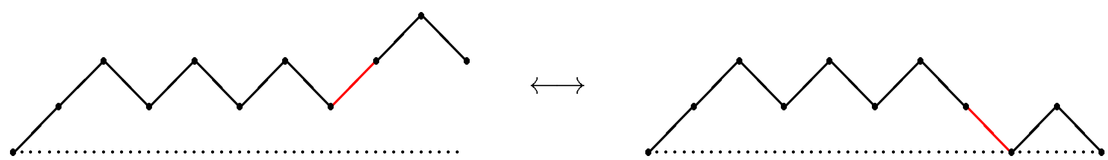


Figure 3

It is graphically obvious what a rotation does to a path in $\mathcal{L}_2$: just transfer the initial step to the end of the path. In $\mathcal{L}_1$ and $\mathcal{L}_3$, to rotate the code sequence of ascent lengths to the left, cut the path into 3 parts: the initial (possibly empty) ascent, the immediately following downstep, and a terminal segment. Then interchange the initial ascent and terminal segment.

5. Proofs In each case, the heart of the matter is showing that ν takes on different values on each of the d cyclic rotations of a code sequence. See [1] for a picture proof in case $d = 2n + 1$ that ν takes on its maximal value exactly once in each cyclic rotation class. (The same picture [1, p. 360] actually shows that ν takes on each of its possible values exactly once.) Exercise: devise similar proofs for the other cases. Note that the lattice paths considered terminate respectively at $y = 0$, $y = 1$ and $y = 2$ in the three cases.

6. Historical Notes Kai-Lai Chung and William Feller [2] proved in 1949 using generating functions that ν is uniformly distributed on $\mathcal{L}_1$ (lattice paths of n upsteps and n downsteps, or coin-tossing games that come out even). This is often referred to as the Chung-Feller theorem, but it was already known in 1908 to Major Percy A. MacMahon [3, p. 168, "a remarkable theorem"] who proved it using formal series (of words on an alphabet). Narayana [4] proved the full assertion for $\mathcal{L}_1$ in 1967. In 1947 Dvoretzky and Motzkin [5] showed ν takes on its maximum value just once in each rotation class in $\mathcal{L}_2$ (this is the basic Cycle Lemma), and I haven't seen $\mathcal{L}_3$ in the literature. For further remarks and generalizations of the Cycle Lemma see [6].

7. Final Remark Of course, none of the three title expressions makes it immediately obvious that C_n is an integer. A fourth method of counting Dyck paths—the André reflection principle of 1887 [7, Chap. 3.1]—does so, expressing C_n as $\binom{2n}{n} - \binom{2n}{n-1}$. For more on an automated method of expressing quantities like those in the title as integer linear combinations of binomial coefficients, see [8].

References

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