Wong's Catalan Number Identity

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The identity [1]

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \left[\frac{n+1-2j}{n+1} \binom{n+1}{j} \right]^2 = \frac{1}{n+1} \binom{2n}{n}$$

has a simple interpretation in terms of lattice paths. The right hand side is the Catalan number C_n, which counts nonnegative lattice paths of n upsteps and n downsteps (nonnegative means they don't dip below the x-axis). The left hand side counts them by where they meet the vertical line $x = n$. The first n steps form a nonnegative path of $n-j$ upsteps and j downsteps for some $j \in [0, \lfloor n/2 \rfloor]$. The number of such paths is $\binom{n}{j}$ [all paths] $- \binom{n}{j-1}$ [paths that touch $y = -1$] $= \frac{n+1-2j}{n+1}\binom{n+1}{j}$. Here, the second count uses André's reflection principle: given a path that touches $y = -1$, flip the initial portion of the path up to the first point of contact with the line $y = -1$ across this line to get a bijection to *all* paths from $(0, -2)$ to the same endpoint, that is, to paths with the same total number n of steps but one fewer downstep. Likewise the last n steps, when reversed, form another such path. The identity follows.

Similarly, the identity $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$ counts all paths of n upsteps and n downsteps by where they meet $x = n$.

References

[1] B. C. Wong, Emma T. Lehmer, American Math Monthly, Volume 37, Issue 10 (Dec., 1930), 558.