INVERSION SEQUENCES AVOIDING A QUADRUPLE OF LENGTH-3 PATTERNS

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Abstract

An inversion sequence is a sequence of integers $e = e_0 \cdots e_n$ which satisfies $0 \leq e_i \leq i$, for all $i = 0, 1, \ldots, n$. For a set of patterns $B$, let $I_n(B)$ be the set of inversion sequences of length $n$ that avoid all the patterns from $B$. We say that two sets of patterns $B$ and $C$ are I-Wilf-equivalent if $|I_n(B)| = |I_n(C)|$, for all $n \geq 0$. In this paper, we show that the number of I-Wilf-equivalences among quadruples of length-3 patterns is at least 212 and at most 215, where three open cases remain.

1. Introduction

Any word $e = e_0 \cdots e_n$ such that $0 \leq e_i \leq i$ for all $i = 0, 1, \ldots, n$ is called an inversion sequence [7, 12] of length $n$. We denote the set of inversion sequences of length $n$ by $I_n$.

We say that a word $u = u_1 \cdots u_n$ is order-isomorphic to a word $v = v_1 \cdots v_n$ if for every pair of indices $i, j \in [n]$, we have $u_i < u_j$ ($u_i = u_j$) if and only if $v_i < v_j$ ($v_i = v_j$). We say that the word $w$ contains a word $p = p_1 \cdots p_k$ if $w$ contains a subsequence of length $k$ which is order-isomorphic to $p$. Otherwise, we say that $w$ avoids $p$. We denote the set of all inversion sequences of length $n$ that avoid a pattern $p$ by $I_n(p)$. For a set of patterns $P$, define $I_n(P) = \cap_{p \in P} I_n(p)$, for all $n \geq 0$. We say that two sets of patterns $P$ and $Q$ are I-Wilf-equivalent, and we write $P \sim I Q$, if $|I_n(P)| = |I_n(Q)|$, for all $n \neq 0$.

Pattern avoidance in inversion sequences was initiated in [7, 12]. Subsequently, several researchers studied the number of I-Wilf-equivalences for single, pairs of, and triples of length-3 patterns:

- Martinez and Savage [14] generalized and extended the notion of pattern-
avoidance for inversion sequences to triples of binary relations that lead to new conjectures and open problems. In particular, some of these results are related to quadruples of length-3 patterns as mentioned in Table 1 (also, see [6]). Most conjectures in this work have been solved in [1, 6, 9, 10], where both generating functions and bijections were employed.

• The results of [2, 5, 14, 15] determined all the I-Wilf-equivalence classes of pairs of length-3 patterns. So, it showed that there are 48 Wilf classes among 78 pairs of length-3 patterns; for a complete list of the classes with open cases in terms of enumeration see Tables 1 and 2 in [15]. Kotsireas, Mansour, and Ydrm [8] enumerated of some of these open cases (see Remark 1).

• Callan, Jehnek, and Mansour [4] showed that the number of I-Wilf-equivalence classes among triples of length-3 patterns is 137, 138 or 139. In particular, it remains to prove $\{101, 102, 110\} \sim \{021, 100, 101\}$ and $\{100, 110, 201\} \sim \{100, 120, 210\}$.

We refer the reader to [4] and references therein. The aim of this paper is to prove the following result.

**Theorem 1.** The number of I-Wilf-equivalence classes among quadruple length-3 patterns is at least 212 and at most 215.

Based on numerical results, Table 2, and Theorem 1, we conjecture the following.

**Conjecture 1.** We conjecture

• Class 152: $\{010, 100, 102, 210\} \sim \{011, 201, 210\}$;

• Class 166: $\{010, 100, 110, 201\} \sim \{010, 101, 120, 201\}$;

• Class 207: $\{100, 101, 110, 201\} \sim \{101, 110, 120, 210\}$.

Note that we checked the conjecture up to $n = 13$ and we commented on these cases in Table 1 as “still Open”. So if we assume that the conjecture is true, then Theorem 1 shows that there are exactly 212 I-Wilf-equivalence classes among quadruple length-3 patterns.

**2. The strategy to prove Theorem 1**

Define $P = \{000, 001, 010, 011, 012, 021, 100, 101, 102, 110, 120, 201, 210\}$ to be the set of all length-3 patterns. The goal of this paper (which we don’t quite achieve) is to prove there are exactly 212 I-Wilf-equivalences among $B \subset P$ with $|B| = 4$. Since the number of subsets $B$ of $P$ with $|B| = 4$ is 715, it seems impossible to reach
our goal by constructing explicit bijections between classes of inversion sequences. The way out is to combine several steps as follows.

First step: We find all the sequences \( \{ |I_n(B)| \}_n \), for all \( B \subset P \) with \( |B| = 4 \). Table 2 in the Appendix below divides the 715 4-subsets of \( P \) into 212 classes, where the first column of this table assigns the number of the class. Theorem 1 is equivalent to proving that the classes in Table 2 are exactly the I-Wilf-equivalences among quadruples of length-3 patterns.

Second step: Let \( C \) be any class in Table 2. We say that \( C \) is trivial if \( C \) contains exactly one subset. Otherwise, \( C \) is nontrivial. Since each trivial class in Table 2 is an I-Wilf-equivalence, we need to consider only the nontrivial classes in Table 2. There are exactly 120 trivial classes, denoted by \( T \) in the first column in Table 2. Thus, it remains to consider \( 212 - 120 = 92 \) nontrivial classes. Table 1 below contains only the 92 nontrivial classes, where in its first column we retain the class number from Table 2.

Third step: Let \( B \) be any set of patterns in \( P \). We say that \( B \) is reducible if there exists \( C \subseteq \subseteq B \) such that \( I_n(B) = I_n(C) \), for all \( n \geq 0 \). In this context we write \( C \sim B \). Clearly, \( C \sim B \) implies \( C \sim B \).

Theorem 2. We have

\[
\begin{align*}
\{001\} & \sim \{001, 101\}, \\
\{001\} & \sim \{001, 102\}, \\
\{001\} & \sim \{001, 201\}, \\
\{011\} & \sim \{011, 101\}, \\
\{011\} & \sim \{011, 110\}, \\
\{012\} & \sim \{012, 102\}, \\
\{012\} & \sim \{012, 120\}, \\
\{021\} & \sim \{021, 201\}, \\
\{021\} & \sim \{021, 210\}.
\end{align*}
\]

Proof. Since the proofs are similar, we show only the equivalence \( \{001\} \sim \{001, 101\} \). Clearly, \( I_n(\{001, 101\}) \subseteq I_n(\{001\}) \). So it remains to show that \( I_n(\{001\}) \subseteq I_n(\{001, 101\}) \). Let \( \pi = \pi_0 \pi_1 \cdots \pi_n \in I_n(\{001\}) \) with \( n \geq 3 \) (clearly, the statement holds for \( n \leq 2 \)) and assume that \( \pi \) contains 101. Thus, there exist \( 0 \leq i < j < k \leq n \) such that \( \pi_i = \pi_k > \pi_j \). By induction we prove that \( \pi_j \geq m \) and \( i \geq m + 1 \), for any \( m = 0, 1, \ldots, n - 3 \).

Clearly, the claim holds for \( m = 0 \). Since \( \pi_0 = 0 \), we have \( i \geq 1 \). If \( \pi_j = 0 \) then \( \pi \) contains 001 (as \( \pi_0 \pi_j \pi_k \)), so \( \pi_i > \pi_j \geq 1 \) and then \( i \geq 2 \). So the claim holds for \( m = 1 \). Assume that the claim holds for \( m \) and let us prove it for \( m + 1 \). Suppose, for a contradiction, that \( \pi \in I_n(\{001\}) \) is such that there exists \( 0 \leq i < j < k \leq n \) with \( \pi_i = \pi_k > \pi_j \geq m \) and \( i \geq m + 1 \). Since \( \pi \) avoids 001, all the letters left of \( \pi_i \) are different, so \( \pi_s = s \) for all \( s = 0, 1, \ldots, m \). If \( \pi_j = m \), then \( \pi \) contains \( \pi_m \pi_j \pi_k = mms \pi_k \), that is, it contains 001. Thus, \( \pi_j \geq m + 1 \). Since \( \pi \) is an inversion sequence and \( \pi_i > \pi_j \), we have \( i \geq m + 2 \). Hence, by induction on \( m \), we have that \( \pi \) satisfies the claim for \( m = n - 3 \), thus \( \pi = 01 \cdots n - 3 \pi_n - 2 \pi_n - 1 \pi_n \) and \( \pi_n > \pi_{n-1} \geq n - 3 \). Since \( \pi \) avoids 001, we have that \( \pi_{n-1} \geq n - 2 \), so \( \pi_{n-2} > n - 2 \),
which contradicts the fact that $\pi_s \leq s$ for all $s = 0, 1, \ldots, n$ ($\pi \in I_n$). Thus, $\pi$ avoids 101, which completes the proof.

In the next theorem, we describe all the subsets of 3 patterns in $P$ reducible to subsets of 2 patterns in $P$ that are not obtained from Theorem 2. Here, we omit the proof.

**Theorem 3.** We have

- $\{000, 001\} \overset{\sim}{\sim} \{000, 001, 100\}$;
- $\{000, 011\} \overset{\sim}{\sim} \{000, 011, \tau\}$, for any $\tau = 100, 201, 210$;
- $\{000, 012\} \overset{\sim}{\sim} \{000, 012, \tau\}$, for any $\tau = 201, 210$;
- $\{000, 021\} \overset{\sim}{\sim} \{000, 021, 100\}$;
- $\{001, 010\} \overset{\sim}{\sim} \{001, 010, \tau\}$, for any $\tau = 021, 100, 110, 120, 210$;
- $\{001, 011\} \overset{\sim}{\sim} \{001, 011, 021\}$;
- $\{001, 012\} \overset{\sim}{\sim} \{001, 012, 021\}$;
- $\{001, 110\} \overset{\sim}{\sim} \{001, 110, 210\}$;
- $\{001, 120\} \overset{\sim}{\sim} \{001, 120, 210\}$;
- $\{010, 011\} \overset{\sim}{\sim} \{010, 011, 100\}$;
- $\{010, 012\} \overset{\sim}{\sim} \{010, 012, \tau\}$, for any $\tau = 101, 201$;
- $\{010, 021\} \overset{\sim}{\sim} \{010, 021, \tau\}$, for any $\tau = 100, 101, 102, 110, 120$.

In the next theorem, we describe all the reducible subsets of 4 patterns in $P$ to subsets of 3 patterns in $P$ that are not obtained from Theorems 2 and 3. Again, we omit the proof.

**Theorem 4.** We have

- $\{001, 011, 012\} \overset{\sim}{\sim} \{001, 011, 012, 210\}$;
- $\{001, 011, 100\} \overset{\sim}{\sim} \{001, 011, 100, 210\}$;
- $\{001, 012, 100\} \overset{\sim}{\sim} \{001, 012, 100, 210\}$.

Note that when we have a reducible subset of four patterns in $P$, we can consider the references [4, 14, 15]. These references considered the I-Wilf-equivalences and enumerations of $|I_n(B)|$ whenever $B$ is any pair or triple of patterns from $P$. In Table 1, for given $B$ in a class $C$, if the computations for $I_n(B)$ are not simple, we write $B = B'$ and cite Theorems 2-4, where $B$ is reducible to $B'$. 
Fourth step: First, following [8], we define the generating tree (see [16]) $\mathcal{T}(B)$. Set $I_B = \bigcup_{n=0}^{\infty} I_n(B)$. The tree $\mathcal{T}(B)$ is understood to be empty if there is no inversion sequence of arbitrary length avoiding the set $B$, that is, $0 \in B$. Otherwise, the root can always be taken as 0. Starting with this root which stays at level 0, we construct the remainder of the nodes of the tree $\mathcal{T}(B)$ as follows: the children of $e_0e_1\cdots e_{n-1} \in I_{n-1}(B)$ are obtained from the set $\{e_0e_1\cdots e_{n-1}e_n \mid e_n = 0, 1, \ldots, n\}$ by obeying the pattern-avoiding restrictions of the patterns in $B$.

We define an equivalence relation on nodes of $\mathcal{T}(B)$. Let $\mathcal{T}(B; e)$ be the subtree consisting of the inversion sequence $e$ as the root and its descendants in $\mathcal{T}(B)$. We say that $e$ is equivalent to $e'$ if and only if $\mathcal{T}(B; e) \equiv \mathcal{T}(B; e')$ (in the sense of plain trees). Let $\mathcal{T}'(B)$ be the same tree $\mathcal{T}(B)$ where we replace each node $e$ by the first node $e' \in \mathcal{T}(B)$ from top to bottom and from left to right in $\mathcal{T}(B)$ such that $\mathcal{T}(B; e) \equiv \mathcal{T}(B; e')$. From now on, we identify $\mathcal{T}'(B)$ with $\mathcal{T}(B)$.

We are now ready to describe the details of the fourth step which is based on the algorithm of [8]. Let $C$ be any nontrivial class in Table 2. For each subset $B \subseteq C$, we run the main algorithm of [8], call it Algorithm KMY, for guessing and proving (if possible) the rules of the generating tree $\mathcal{T}(B)$. Then, we translate these rules into a system of equations and we solve for $F_B(x) = \sum_{n \geq 0} |I_n(B)|x^{n+1}$. For examples, we refer the reader to [4, 8, 13]. See Table 1 for all the generating trees $\mathcal{T}(B)$ that we obtained and all the corresponding generating functions $F_B(x) = \sum_{n \geq 0} |I_n(B)|x^{n+1}$.

**Remark 1.** In [8], Kotsireas, Mansour, and Ydrm suggested an algorithm for guessing and proving (if possible) the rules of the generating tree $\mathcal{T}(B)$. In particular, they solved six open cases (see Tables 1 and 2 in [15]) for such pattern classes: $I_n(000,021)$, $I_n(102,021)$, $I_n(100,012)$, $I_n(120,210)$, Wilf-equivalent $I_n(011,201)$ and $I_0(011,210)$, and Wilf-equivalent $I_n(100,021)$ and $I_n(110,021)$. Moreover, they extended the algorithm to the case of restricted growth sequences (for pattern avoidance on restricted growth sequences, see [11]) and presented an explicit formula for the generating function for the number of restricted growth sequences of length $n$ that avoid either $\{12313,12323\}$, $\{12313,12323,12333\}$, or $\{123\cdots \ell 1\}$.

We end this section with Table 1 which presents the 92 nontrivial classes (see second column). In the third and fourth columns of the table, we present the rules of the generating tree $\mathcal{T}(B)$ (the root is always 0) and the corresponding generating function $F_B(x)$, whenever $B$ is any set in the first column of the table.

<table>
<thead>
<tr>
<th>Class</th>
<th>Rules of $\mathcal{T}(B)$</th>
<th>$F_B(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${000,012,024,034}$</td>
<td>$x + 2x^2 + x^3$</td>
</tr>
<tr>
<td>2</td>
<td>${000,000,002,010,021}$</td>
<td>$x + 2x^2 + x^3$</td>
</tr>
<tr>
<td></td>
<td>${002,010,012,012}$</td>
<td>$x + 2x^2 + x^3$</td>
</tr>
<tr>
<td>Class</td>
<td>B</td>
<td>Rules of $T^r(B)$</td>
</tr>
<tr>
<td>-------</td>
<td>---</td>
<td>-----------------</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>( 000,001,012,021 )</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>[ 001,011,012 \sub{100} ]</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
<td>[ 001,010,012 \sub{100} ]</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>[ 001,011,012 \sub{201} ]</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>[ 001,010,012 \sub{201} ]</td>
</tr>
<tr>
<td>9</td>
<td>15</td>
<td>[ 001,011,012 \sub{201} ]</td>
</tr>
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<td>10</td>
<td>15</td>
<td>[ 001,010,012 \sub{201} ]</td>
</tr>
<tr>
<td>11</td>
<td>15</td>
<td>[ 001,011,012 \sub{201} ]</td>
</tr>
<tr>
<td>12</td>
<td>15</td>
<td>[ 001,010,012 \sub{201} ]</td>
</tr>
<tr>
<td>13</td>
<td>15</td>
<td>[ 001,011,012 \sub{201} ]</td>
</tr>
<tr>
<td>14</td>
<td>15</td>
<td>[ 001,010,012 \sub{201} ]</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>[ 001,011,012 \sub{201} ]</td>
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Continuation of Table 1
<table>
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<tr>
<th>Class</th>
<th>Rules of $T'(b)$</th>
<th>Dominant of Table 1</th>
<th>$P_{g}(x)$</th>
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<tr>
<td>16</td>
<td>$a_m \rightarrow (00)_m+1$</td>
<td>$x \rightarrow 2x^2 + 3x^3 + 4x^4$</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>$a_0 \rightarrow b_0, a_1 \rightarrow b_1^m, a_m \rightarrow b_m^m, a_{m+1} \rightarrow b_m, b_m \rightarrow b^m$, where</td>
<td>$x \rightarrow (x+1)^3 - 1$</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>$a_m \rightarrow a_{m+1}, b_0, \ldots, b_m; b_m \rightarrow b_1, \ldots, b_{m-1},$ where</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>$a_m \rightarrow 0^m, b_m \rightarrow a_m^m$</td>
<td>$x \rightarrow (1-x)^2 + 2 + 3$</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>$a_0 \rightarrow 0, 00 \rightarrow (00)_2; 01 \rightarrow 01$</td>
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### Table 1: Continuation of Table 1

<table>
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<th>Class</th>
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<tr>
<td>36</td>
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<td>See Subsection 4.1</td>
</tr>
<tr>
<td>37</td>
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<td>See Subsection 4.2</td>
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<tr>
<td>38</td>
<td></td>
<td>See Subsection 4.3</td>
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<td>39</td>
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<td>$x + 2x^2 + 4x^3 + 3x^4$</td>
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<td>$x + 2x^2 + 4x^3 + 4x^4$</td>
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<td>45</td>
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<td>$x + 2x^2 + 3x^3 + 5x^4 + x^5$</td>
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<td>46</td>
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<tr>
<td>47</td>
<td></td>
<td>$x + 2x^2 + 3x^3 + 5x^4 + x^5$</td>
</tr>
<tr>
<td>48</td>
<td></td>
<td>$x + 2x^2 + 3x^3 + 5x^4 + x^5$</td>
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</tbody>
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*Note: The table continues with similar entries.*
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<th>Class</th>
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<th>( P_2(s) )</th>
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<td>([001.012.010.210] )</td>
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<td>([010.012.010.102] )</td>
<td>( 0 = 00, 01, 00 = 00, 01 = 00, 01 = 00, 01 = 00, 01 = 00 )</td>
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<td>( \frac{x(1-x+x^2)}{1-x} )</td>
</tr>
</tbody>
</table>

\( a_m \rightarrow a_{m+1} \cdot (01)^m \cdot 01 = 01, a_m = 0^m \)
<table>
<thead>
<tr>
<th>Class</th>
<th>Rule of F′(h)</th>
<th>( P_F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[010,012,010,120]</td>
<td>( a_m \rightarrow a_{m+1}, b_1 \ldots b_m ); ( b_m \rightarrow b_1, \ldots, b_m ), where ( a_m = 0^m ); ( b_m = a_m m )</td>
<td>( \frac{x}{1-x^2} )</td>
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<td>[010,012,100,120] ( \L_0 )</td>
<td>( 010 ), 011, 101, 110 - Theorems 2-3</td>
<td>See [4]</td>
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<tr>
<td>[000,010,021,101]</td>
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</tr>
<tr>
<td>[012,021,101,110] ( \L_0 )</td>
<td>( 010, 011 ) - Theorems 2-3</td>
<td>See [4]</td>
</tr>
<tr>
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</tr>
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<td>Class</td>
<td>Rule of $F_{14}(x)$</td>
<td>Notes</td>
</tr>
<tr>
<td>-------</td>
<td>---------------------</td>
<td>-------</td>
</tr>
<tr>
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<td>Theorem 2 See [4]</td>
</tr>
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</tr>
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<td>$\leq (012, 120, 210)$</td>
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<tr>
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<td>$\leq (011, 010, 120)$</td>
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<tr>
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<td>$\leq (000, 021, 102)$</td>
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</tr>
</tbody>
</table>

Continued in Table 1...

\[
\begin{align*}
\text{11} & \quad \leq (012, 110, 210) & \text{Theorem 2} \\
\text{112} & \quad \leq (012, 100, 201) & \text{Theorem 2} \\
\text{115} & \quad \leq (012, 110, 201) & \text{Theorem 2} \\
\text{118} & \quad \leq (000, 021, 102) & \text{Theorem 2} \\
\text{121} & \quad \leq (000, 021, 102) & \text{Theorem 2} \\
\text{124} & \quad \leq (000, 021, 102) & \text{Theorem 2} \\
\text{127} & \quad \leq (000, 021, 102) & \text{Theorem 2} \\
\end{align*}
\]
### Theorem 11

Theorem 11

\[ 010, 100, 120 \]

\[ 010, 110, 120 \]

\[ 010, 101, 201 \]

\[ 011, 100, 201 \]

\[ 011, 101, 201 \]

\[ 011, 102, 201 \]

\[ 011, 110, 201 \]

\[ 011, 111, 201 \]

\[ 011, 102, 210 \]

\[ 011, 103, 210 \]

\[ 011, 112, 210 \]

\[ 011, 113, 210 \]

\[ 011, 120, 210 \]

\[ 011, 121, 210 \]

\[ 011, 122, 210 \]

\[ 011, 123, 210 \]

\[ 012, 100, 201 \]

\[ 012, 101, 201 \]

\[ 012, 102, 201 \]

\[ 012, 103, 201 \]

\[ 012, 110, 201 \]

\[ 012, 111, 201 \]

\[ 012, 112, 201 \]

\[ 012, 113, 201 \]

\[ 012, 120, 201 \]

\[ 012, 121, 201 \]

\[ 012, 122, 201 \]

\[ 012, 123, 201 \]

\[ 013, 100, 201 \]

\[ 013, 101, 201 \]

\[ 013, 102, 201 \]

\[ 013, 103, 201 \]

\[ 013, 110, 201 \]

\[ 013, 111, 201 \]

\[ 013, 112, 201 \]

\[ 013, 113, 201 \]

\[ 013, 120, 201 \]

\[ 013, 121, 201 \]

\[ 013, 122, 201 \]

\[ 013, 123, 201 \]

\[ 014, 100, 201 \]

\[ 014, 101, 201 \]

\[ 014, 102, 201 \]

\[ 014, 103, 201 \]

\[ 014, 110, 201 \]

\[ 014, 111, 201 \]

\[ 014, 112, 201 \]

\[ 014, 113, 201 \]

\[ 014, 120, 201 \]

\[ 014, 121, 201 \]

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\[ 014, 123, 201 \]

\[ 015, 100, 201 \]

\[ 015, 101, 201 \]

\[ 015, 102, 201 \]

\[ 015, 103, 201 \]

\[ 015, 110, 201 \]

\[ 015, 111, 201 \]

\[ 015, 112, 201 \]

\[ 015, 113, 201 \]

\[ 015, 120, 201 \]

\[ 015, 121, 201 \]

\[ 015, 122, 201 \]

\[ 015, 123, 201 \]

\[ 016, 100, 201 \]

\[ 016, 101, 201 \]

\[ 016, 102, 201 \]

\[ 016, 103, 201 \]

\[ 016, 110, 201 \]

\[ 016, 111, 201 \]

\[ 016, 112, 201 \]

\[ 016, 113, 201 \]

\[ 016, 120, 201 \]

\[ 016, 121, 201 \]

\[ 016, 122, 201 \]

\[ 016, 123, 201 \]
3. Proof of Theorem 1

Let $B$ be any set of patterns and let $T(B)$ be the generating tree for the class $I_B$. The length of a node $v \in T(B)$ is defined to be the number of letters in $v$. For any $k \geq 1$, let $D_k(B)$ be the multiset of all nodes of length $k$ at level $k-1$ in $T(B)$. For each node $v \in D_k(B)$, we denote the multiset of all children of $v$ at level $k$ in $T(B)$ by $N_k(B; v)$. A generating tree $T(B)$ is said to be $d$-regular (see [13]) if there exists $k \geq 1$ such that

- the number of different nodes in $D_r(B)$ equals $d$, for all $r > k$;
- for any $v \in D_r(B)$ and $w \in N_r(B; v)$, the number of occurrences of $w$ in $N_r(B; v)$ does not depend on $r$, whenever $r > k$.

Clearly, $T(B)$ is 0-regular if and only if the set of all nodes of the generating tree $T(B)$ is finite. For instance, the generating tree

$$0^n \leadsto (0^{n+1})(01) \cdots (0^m 1),$$

$$0^m 1 \leadsto (01)(021)^{m-1}(0^m 1),$$

$$010 \leadsto 010$$

Continuation of Table 1

<table>
<thead>
<tr>
<th>Class</th>
<th>Rule of $T(B)$</th>
<th>$B_r$</th>
<th>$P_g(C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>196</td>
<td>${020,100,101,102,200}$</td>
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</tr>
<tr>
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<tr>
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<td>${020,100,101,102,200}$</td>
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</tr>
</tbody>
</table>

End of Table 1
is 2-regular. But the generating tree

\[
0^m \rightarrow (0^{m+1})(0^m 1) \cdots (0^m m),
\]

\[
0^m j \rightarrow (0^{m+1})(0^m(j+1)) \cdots (0^m m), \quad 1 \leq j \leq m
\]

and the generating tree

\[
0^m \rightarrow (0^{m+1})(01) \cdots (0^m 1),
\]

\[
0^m 1 \rightarrow (0^m 1)^m(01)^2
\]

are not \(d\)-regular, for any \(d\). Several examples of \(d\)-regular generating trees are presented in [4, 8, 13] and [13] contains a five step algorithm to obtain an explicit formula for the generating function \(F_B(x)\) from a \(d\)-regular generating tree \(T(B)\). Accordingly, in this paper we omit the proof details for the generating function \(F_B(x)\) whenever the generating tree \(T(B)\) is \(d\)-regular for some \(d\). So, we only consider the classes \(C\) in Table 1 such that there exists \(B \in C\) such that \(T(B)\) is not \(d\)-regular, for any \(0 \leq d \leq 6\).

**Theorem 5 (Class 63).** We have \(
\{010, 011, 120, 201\} \sim \{010, 011, 120, 210\}\). Moreover, we have

\[
F_{\{010,011,120,201\}}(x) = \frac{x}{1-x} - \sum_{j \geq 1} \frac{(1-x)(1-2x)x^{2j}}{(x^{j+2} + 1 - 2x) \prod_{i=2}^{j+1} (x^i + 1 - 2x) \prod_{i=2}^{j+1} (x^i - (1-2x)^2)}.
\]

**Proof.** Let \(B = \{010, 011, 120, 201\}\) or \(B = \{010, 011, 120, 210\}\). Then, the rules of the generating tree of \(T(B)\) are given by (here we used Algorithm KMY to guess and prove)

\[
a_m \rightarrow a_{m+1}b_{m,1} \cdots b_{m,m},
\]

\[
b_{m,j} \rightarrow b_{m,j-1} \cdots b_{m+2-j,1}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j},
\]

where \(a_m = 0^m\) and \(b_{m,j} = 0^m j\) with \(j = 1, 2, \ldots, m\). Define \(A_m(x)\) (respectively, \(B_{m,j}(x)\)) to be the generating function for the number of nodes at level \(n \geq 1\) for the subtree of \(T(B; a_m)\) (respectively, \(T(B; b_{m,j})\)), where its root stays at level 0. Thus,

\[
A_m(x) = x + xA_{m+1}(x) + x \sum_{j=1}^{m} B_{m,j}(x),
\]

\[
B_{m,j}(x) = x + x \sum_{i=1}^{j-1} B_{m+i-j+1,i}(x) + x \sum_{i=1}^{m+1-j} B_{m+1-j,i}(x).
\]
Define $B_m(u) = \sum_{j=1}^{m} B_{m,j}(x) u^{m-j}$, $A(v) = \sum_{m \geq 1} A_m(x) v^{m-1}$, and $B(v,u) = \sum_{m \geq 1} B_m(u) v^{m-1}$. Then, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v} (A(v) - A(0)) + xB(v,1), \quad (1)$$

$$B(v,u) = \frac{x}{(1-v)(1-uv)} + \frac{x}{u(1-v)} (B(v,u) - B(v,0)) + \frac{x}{1-v} B(vu,1). \quad (2)$$

By substituting $u = x/(1-v)$ into (2), we have

$$B(v,0) = \frac{x}{1-v-xv} + \frac{x}{1-v} B(xv/(1-v),1).$$

Thus, by substituting $u = 1$ into (2) and then solving for $B(v,1)$, we have

$$B(v,1) = \frac{x(1-x-v)}{(1-v)(1-v-xv)(1-2x-v)} - \frac{x^2}{(1-v)(1-2x-v)} B(xv/(1-v),1).$$

By iterating this equation indefinitely, we obtain for $|x| < 1$,

$$B(v,1) = -\sum_{j \geq 0} \frac{(1-x)(1-x-v)x^{2j+1} \prod_{i=0}^j (vx^i - v - x + 1)^2}{(vx^{j+2} + 1 - v - x) \prod_{i=0}^{j+1} (vx^i + 1 - v - x) \prod_{i=0}^{j+1} (vx^i - (1-2x)(1-x-v))}.$$ 

Hence, by taking $v = x$, (1) gives $A(0) = \frac{x}{1-x} + xB(x,1)$, which completes the proof. \hfill \square

**Theorem 6 (Class 82).** We have

$$F_{\{011,100,102,120\}}(x) = \frac{x(1-x-x^2)}{(1+x)(1-2x)}.$$

**Proof.** Let $B = \{011,100,102,120\}$. Then, the rules of the generating tree of $T(B)$ is given by

- $a_m \leadsto a_{m+1} b_{m,1} \cdots b_{m,m}$,
- $b_{m,1} \leadsto c_{1,0} d_{m} b_{m-1,1} \cdots b_{m-1,m-1}$,
- $b_{m,j} \leadsto c_{j,0} \cdots c_{j,j-2} c_{j,0} d_{m+1-j} b_{m-j,1} \cdots b_{m-j,m-j}$, \hspace{1em} $j = 2, 3, \ldots, m-1$,
- $b_{m,m} \leadsto c_{m,0} \cdots c_{m,m-2} c_{m,0} d_1$,
- $c_{m,1} \leadsto c_{m-2,0} \cdots c_{m-2,m-4} c_{m-2,0}$,
- $c_{m,j} \leadsto c_{j,0} \cdots c_{j,j-2} c_{j,0} c_{m-1-j,0} \cdots c_{m-1-j,m-3} c_{m-1-j,0}$, \hspace{1em} $j = 1, 2, \ldots, m-3$,
- $c_{m,m-2} \leadsto c_{m-2,0} \cdots c_{m-2,m-4} c_{m-2,0} c_{1,0}$,
- $d_m \leadsto d_{m} b_{m-1,1} \cdots b_{m-1,m-1}$,
- $c_{1,0} \leadsto c_{1,0}$. 
Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_{m,j}(x)$, $D_m(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $T'(B; a_m)$ (respectively, $T'(B; b_{m,j})$, $T'(B; c_{m,j})$, $T'(B; d_m)$), where its root stays at level 0. Thus,

\[
A_m(x) = x + xA_{m+1}(x) + x \sum_{i=1}^{m} B_{m,i}(x),
\]

\[
B_{m,j}(x) = x + x \left( C_{j,0} + \sum_{i=0}^{j-2} C_{j,i}(x) \right) + xD_{m+1-j}(x) + x \sum_{i=1}^{m-j} B_{m-j,i}(x),
\]

\[
C_{m,j}(x) = x + xC_{1,0}(x)(\delta_{j=1} + \delta_{j=m-2}) + x \left( c_{m-1-j,0} + \sum_{i=0}^{m-3-j} C_{m-1-j,i}(x) \right)
\]

\[
\quad + \left( C_{j,0}(x) + \sum_{i=0}^{j-2} C_{j,i}(x) \right),
\]

\[
D_m(x) = x + xD_m(x) + x \sum_{i=1}^{m-1} B_{m-1,i}(x).
\]

Define $A(v) = \sum_{m \geq 2} A_m(x)v^{m-1}$, $B(v, u) = \sum_{m \geq 1} \sum_{j=1}^{m} B_{m,j}(x)u^{j-1}v^{m-1}$, $C(v, u) = \sum_{m \geq 2} \sum_{j=0}^{m-2} C_{m,j}(x)u^{j}v^{m-1}$, and $D(v) = \sum_{m \geq 1} D_m(x)v^{m-1}$. Then, the above recurrence can be written as

\[
A(v) = \frac{x}{1 - v} + \frac{x}{v}(A(v) - A(0)) + xB(v, 1),
\]

\[
D(v) = \frac{x}{1 - v} + xD(v) + xvB(v, 1),
\]

\[
B(v, u) = \frac{x}{(1 - v)(1 - uv)} + \frac{x}{1 - v}(C(uv, 1) + C(uv, 0))
\]

\[
\quad + \frac{x}{1 - uv}D(v) + \frac{xv}{1 - uv}B(v, 1),
\]

\[
C(v, u) = \frac{xv}{(1 - v)(1 - uv)} + \frac{x^2v}{1 - uv}
\]

\[
\quad + \frac{x^2v^2}{1 - v} + \frac{xv}{1 - uv}(C(v, 1) + C(v, 0)) + \frac{xv^2u}{1 - v}(C(uv, 1) + C(uv, 0)).
\]

By taking either $u = 1$ or $u = 0$ into (6), we have a system of equations with variables $C(v, 1)$ and $C(v, 0)$. By solving this system, we have

\[
C(v, 1) = \frac{(x + 1)vx}{(1 - v)(1 - x - 2xv)}, \quad C(v, 0) = \frac{(1 + x - v - 2vx)vx}{2v^2x + v^2 - 2vx - 2v + 1}.
\]

By solving (4) for $D(v)$, we have

\[
D(v) = \frac{x}{(1 - v)(1 - x)} + \frac{xv}{1 - x}B(v, 1).
\]
By substituting expressions of $C(v,0)$, $C(v,1)$, and $D(v)$ into (5) with $u = 1$, and then solving for $B(v,1)$, we obtain

$$B(v,1) = \frac{x(2vx + x^2 + v - x - 1)}{(2vx^2 + v^2 - 2vx - 2v + 1)(v + x - 1)}$$

Then, by taking $u = x$ into (3) with using expression of $B(v,1)$, we complete the proof. 

By similar arguments as in the proof of Theorem 6, we obtain the following result.

**Theorem 7 (Class 90).** We have

$$F_{[011,102,120,201]}(x) = F_{[011,102,120,210]}(x) = \frac{x(1 - 2x + 2x^2)}{(1 - x)^2(1 - 2x)}.$$ 

Moreover, the rules of the generating tree $\{011,102,120,201\}$ are given by

- $a_m \rightarrow a_{m+1}b_{m,1} \cdots b_{m,m}$,
- $b_{m,j} \rightarrow d_1^j c_{m+1-j,2} \cdots c_{m+1-j,m+2-j}$, $j = 1,2$,
- $b_{m,j} \rightarrow d_3^2 b_{1,1} d_1 \cdots d_j c_{m+1-j,2} \cdots c_{m+1-j,m+2-j}$, $j = 3,4,\ldots,m-1$,
- $b_{m,m} \rightarrow d_1^3 b_{1,1} d_1 \cdots d_m$,
- $c_{m,j} \rightarrow e_3 \cdots e_j c_{m+2-j,2} \cdots c_{m+2-j,m+3-j}$, $j = 2,3,\ldots,m$,
- $c_{m,m+1} \rightarrow e_3 \cdots e_m d_1$,
- $d_m \rightarrow d_1^2 b_{1,1} d_1 \cdots b_{m-1}$,
- $e_m \rightarrow e_3 \cdots e_{m-1}$,

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$, $d_m = a_m m(m-1)$, and $e_m = a_m 1m(m-1)$. The rules of the generating tree $\{011,102,120,210\}$ are given by

- $a_m \rightarrow a_{m+1} b_{m,1} \cdots b_{m,m}$,
- $b_{m,j} \rightarrow d_1 e_2 \cdots e_j c_{m+1-j,2} \cdots c_{m+1-j,m+2-j}$, $j = 1,2,\ldots,m-1$,
- $b_{m,m} \rightarrow d_m e_2 \cdots e_m d_3$,
- $c_{m,j} \rightarrow e_2 \cdots e_j c_{m+2-j,2} \cdots c_{m+2-j,m+3-j}$, $j = 2,3,\ldots,m$,
- $c_{m,m+1} \rightarrow e_2 \cdots e_m d_1$,
- $d_m \rightarrow d_m e_2 \cdots e_m$,
- $e_m \rightarrow e_2 \cdots e_{m-1}$,

where $a_m = 0^m$, $b_{m,j} = a_m j$, $c_{m,j} = a_m 1j$, $d_m = a_m m0$, and $e_m = a_m m1$. 


Theorem 8 (Class 115). We have \( \{011, 100, 120, 201\} \not\sim \{011, 100, 120, 210\} \).

Moreover,

\[
F_{\{011, 100, 120, 201\}}(x) = \frac{x}{1 - x} + \sum_{j \geq 0} \frac{x^{j+4}(1 + x)^j}{(v_j + 2x - 1)(1 - v_j)^2} \prod_{i=0}^{j-1} (1 - v_i)(v_i + 2x - 1)(v_i x + v_i - 1)
\]

with \( v_j = \frac{(1-x)x^{j+1}}{1-2x+x^{j+1}} \).

Proof. The rules of the generating trees \( T(\{011, 100, 120, 201\}) \) and \( T(\{011, 100, 120, 210\}) \) are given by

\[
a_m \Rightarrow a_{m+1}b_{m,1} \cdots b_{m,m},
\]

\[
b_{m,j} \Rightarrow c_{m+1-j}c_{m-j}b_{m+2-j,1} \cdots b_{m,j-1}b_{m-j,1} \cdots b_{m-j,m-j},
\]

\[
b_{m,m} \Rightarrow c_1b_{2,1} \cdots b_{m-m-1}(012),
\]

\[
c_m \Rightarrow c_mb_{m,1} \cdots b_{m,m},
\]

\[
012 \Rightarrow 012,
\]

where \( a_m = 0^m, b_{m,j} = a_m j \) with \( j = 1, 2, \ldots, m \), and \( c_m = a_m 10 \). Define \( A_m(x) \) (respectively, \( B_{m,j}(x), C_m(x) \)) to be the generating function for the number of nodes at level \( n \geq 1 \) for the subtree of \( T(B; a_m) \) (respectively, \( T(B; b_{m,j}), T(B; c_m) \)), where its root stays at level 0. Thus,

\[
A_m(x) = x + xA_{m+1} + x \sum_{j=1}^{m} B_{m,j}(x),
\]

\[
C_m(x) = x + xC_{m+1} + x \sum_{j=1}^{m} B_{m,j}(x),
\]

\[
B_{m,j}(x) = x + xC_{m+1-j}(x) + xC_{m-j}(x) + x \sum_{i=1}^{m-j} B_{m-j,i}(x) + x \sum_{i=1}^{j-1} B_{m+1-j+i,i}(x),
\]

\[
B_{m,m}(x) = x + xC_1(x) + x \sum_{i=1}^{m-1} B_{1,i+1}(x) + \frac{x^2}{1 - x},
\]

where \( 1 \leq j \leq m - 1 \). Define \( F(v) = \sum_{m \geq 1} F_m(x)v^{m-1} \) where \( F \in \{A, C\} \) and \( B(v, u) = \sum_{m \geq 1} \sum_{j=1}^{m} B_{m,j}(x)u^{m-j}v^{m-1} \). Then, the above recurrence can be
Then

\[ B(v,u) = \frac{x}{(1-v)(1-vu)} + \frac{x^2}{(1-x)(1-v)} + \frac{x}{1-v} \left( C(vu) + vuC(vu) + vuB(vu,1) \right) \]
\[ + \frac{x}{u(1-v)}(B(v,u) - B(v,0)). \]

By third equation with \( u = 1 \), we obtain

\[ B(v,0) = -\frac{(v^2 + 2xv - 2v + 2x + 1)B(v,1) - x^2 - x}{x(1-x)(1-v)}. \]

From second equation, we express the function \( C(v) \) in terms of \( B(v,1) \). Hence, the third equation with \( u = vx/(1-x) \) gives

\[ B(v,1) = a(v)B(vx/(1-v),1) + b(v), \tag{7} \]

where \( a(v) = \frac{x^3}{(x+2z-1)(1-v)^2} \) and \( b(v) = -\frac{x(x+1)(x-1+v)}{(1-v)(x+2z-1)(x+1-x)v}). \) By iterating (7) infinite number of times (here we assumed \( |x| < 1 \), we have

\[ B(v,1) = \sum_{j \geq 0} a(vx^j/(1-v(1-x^j)/(1-x))) \prod_{i=0}^{j-1} b(vx^i/(1-v(1-x^i)/(1-x))), \]

which, by equation of \( A(v) \) with \( v = x \), implies \( A(0) = \frac{x}{1-x} + xB(x,1) \), which completes the proof. \( \square \)

**Theorem 9 (Class 169).** We have \( \{010,100,101,201\} \sim \{010,100,101,210\} \).

**Proof.** Let \( T \) be the tree with a root \( a_1 \) and satisfies the following results

\[ a_m \sim a_{m+1}a_m b_{m,2} \cdots b_{m,m}, \]
\[ b_{m,j} \sim c_{m+2-j,2} \cdots c_{m,j} b_{m+1,j} b_{m,j} \cdots b_{m,m}, \quad 2 \leq j \leq m, \]
\[ c_{m,2} \sim a_m b_{m,2} \cdots b_{m,m}, \]
\[ c_{m,j} \sim c_{m+3-j,2} \cdots c_{m,j-1} b_{m,j-1} \cdots b_{m,m}, \quad 3 \leq j \leq m. \]

Then \( T(\{010,100,101,201\}) \) is given by \( T \) with \( a_m = 0^m, b_{m,j} = a_m, \) and \( c_{m,j} = a_m(j-1) \) and the tree \( T(\{010,100,101,210\}) \) is given by \( T \) with \( a_m = 0^m, b_{m,j} = a_m, \) and \( c_{m,j} = a_m j1 \). Hence, \( \{010,100,101,201\} \sim \{010,100,101,210\} \). \( \square \)
Remark 2. Let $B = \{010, 100, 101, 201\}$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $T(B; a_m)$ (respectively, $T(B; b_{m,j})$), where its root stays at level 0. Moreover, let $A(v) = \sum_{m \geq 1} A_m(x)v^{m-1}$, $B(v, u) = \sum_{m \geq 2} \sum_{j=2}^{m} B_{m,j}(x)u^{m-j}v^{m-2}$, and $C(v, u) = \sum_{m \geq 2} \sum_{j=2}^{m} C_{m,j}(x)u^{m-j}v^{m-2}$. Then, the rules of the generating tree $T(B)$ in the proof of Theorem 9 imply

\[
A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + vA(v) + xB(v, 1),
\]

\[
B(v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{1-v} C(v, u) + \frac{x}{uv}(B(v, u) - B(v, 0)) + \frac{x}{1-u} (B(v, u) - uB(1, u)),
\]

\[
C(v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{uv}(A(uv) - A(0)) + xB(uv, 1) + \frac{x}{u(1-v)} (C(v, u) - C(v, 0)) + \frac{x}{u(1-u)} (B(v, u) - uB(1, u)) - \frac{x}{u} B(v, 0).
\]

Here, we failed to derive from these equations an explicit formula for $A(0)$.

Theorem 10 (Class 172). We have $\{010, 100, 201, 210\} \sim \{010, 101, 201, 210\}$. Moreover,

\[
F_{\{010, 100, 201, 210\}}(x) = \frac{K(x)}{1-K(x)},
\]

where

\[
K(x) = \frac{1}{4}(1+2x-x^2+(1-x)\sqrt{x^2-6x+1}) - \frac{1}{2\sqrt{2}} \sqrt{(x+1)(x^2-4x+1)(x^2-6x+1+x^4-6x^3+4x^2-6x+1)}.
\]

Proof. By Table 1 (Class 172), we see that the generating tree $T(\{010, 100, 201, 210\})$ is the same as the generating tree $T(\{010, 101, 201, 210\})$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $T(B; a_m)$ (respectively, $T(B; b_{m,j})$), where its root stays at level 0. Thus,

\[
A_m(x) = x + xA_{m+1} + xA_m(x) + x \sum_{j=1}^{m} B_{m,j}(x),
\]

\[
B_{m,j}(x) = x + (j-1)xA_{m+2-j}(x) + xB_{m+1,j}(x) + x \sum_{i=j}^{m} B_{m,i}(x),
\]

where $2 \leq j \leq m$. 
Define $A(v) = \sum_{m \geq 1} A_m(x)v^{m-1}$ and $B(v, u) = \sum_{m \geq 2} \sum_{j=2}^{m} B_{m,j}(x)u^{m-j}v^{m-2}$. Then, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xA(v) + xB(v, 1), \quad (8)$$

$$B(v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-v)^2}(A(vu) - A(0))$$

$$+ \frac{x}{uv}(B(v, u) - B(v, 0)) + \frac{x}{1-u}(B(v, u) - uB(vu, 1)). \quad (9)$$

In order to solve (8)-(9), we assume that the generating functions $A(v)$ and $B(v, u)$ satisfy one extra equation

$$B(v, 1) = \frac{A(v)}{1 - v - \frac{x}{2}(1 - \sqrt{1 - 6x + x^2})}. \quad (10)$$

Note that we guessed (10) by looking at the first terms of the generating function $A(v)/B(v, 1)$. Hence, by (8) and (10), we have

$$\left(1 - x - \frac{x}{v} - \frac{2xv}{1 - 2v + x + \sqrt{x^2 - 6x + 1}}\right)A(v) = \frac{x}{1 - v} - \frac{x}{v}A(0).$$

By taking $v = v_0 = K(x)$, we obtain $A(0) = \frac{v_0}{1-v_0}$, which, by using same equation, implies

$$A(v, 1) = \frac{x(v_0 - v)(1 - 2v + x + \sqrt{x^2 - 6x + 1})}{(1 - v)(1 - v_0)((xv - x^2 - 6x + 1) + (v + 1)x^2 + (1 - 2v)x + 2v^2 - v)}.$$ 

Thus, by (10), we have

$$B(v, 1) = \frac{2x(v_0 - v)}{(1 - v)(1 - v_0)((xv - x^2 - 6x + 1) + (v + 1)x^2 + (1 - 2v)x + 2v^2 - v)}.$$ 

Now, by (9), we have

$$\left(1 - \frac{x}{uv} - \frac{x}{1-u}\right)B(v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-v)^2}(A(vu) - A(0))$$

$$- \frac{x}{uv}B(v, 0) - \frac{xu}{1-u}B(vu, 1).$$

By taking $u = u_0 = \frac{x+v-xv^2+\sqrt{(v-1)^2x^2-2vx+(1-2x)x^2}}{2v}$, we obtain

$$B(v, 0) = \frac{vu_0}{(1-v)(1-vu_0)} + \frac{1}{(1-v)^2}(A(vu_0) - A(0)) - \frac{vu_0^2}{1-u_0}B(vu_0, 1),$$

which leads to

$$B(v, u) = \frac{x}{(1-v)(1-vu)} + \frac{x}{vu(1-v)^2}(A(vu) - A(0)) - \frac{x}{u}B(v, 0) - \frac{xu}{1-u}B(vu, 1)$$

$$+ \frac{x}{1-u} - \frac{x}{1-v}.$$
where we do not present the explicit expressions for $B(v,0)$ and $B(v,u)$ because they are too long.

Since $A(v)$ and $B(v,u)$ satisfy (8)-(10), this completes the proof. □

**Theorem 11 (Class 189).** We have $\{100, 102, 120, 201\} \sim \{102, 110, 120, 201\}$. Moreover,

$$F_{\{100,102,120,201\}}(x) = \frac{1 - 3x - x^4 - (1 - x - 2x^3 - x^4)\sqrt{1 - 4x + 1}}{2x^2(2 + x)(1 - x)^2}.$$

**Proof.** We proceed by finding the generating function for each class.

1. The rules of the generating trees $T(\{100, 102, 120, 201\})$ are given by

   $$a_m \Rightarrow a_{m+1}b_{m,1} \cdots b_{m,m},$$
   $$b_{m,j} \Rightarrow c_1 \cdots c_j b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 1 \leq j \leq m,$$
   $$c_m \Rightarrow d_2 \cdots d_{m-1}c_1,$$
   $$d_m \Rightarrow d_2 \cdots d_{m-1},$$

   where $a_m = 0^m$, $b_{m,j} = a_{m+j}$ with $j = 1, 2, \ldots, m$, $c_m = a_m(m-1)$, and $d_m = a_m(m-1)(m-2)$. Define $A_m(x)$ (respectively, $B_{m,j}(x)$, $C_m(x)$, $D_m(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $T(B;a_m)$ (respectively, $T(B;b_{m,j})$, $T(B;c_m)$, $T(B;d_m)$), where its root stays at level 0. Thus,

   $$A_m(x) = x + xA_{m+1} + x \sum_{j=1}^m B_{m,j}(x),$$
   $$B_{m,j}(x) = x + x \sum_{i=1}^j C_i(x) + xB_{m+1,j}(x) + x \sum_{i=1}^{m+1-j} B_{m+1-j,i}(x),$$
   $$C_m(x) = x + xC_1(x) + x \sum_{i=2}^m D_i(x),$$
   $$D_m(x) = x + x \sum_{i=2}^{m-1} D_i(x),$$

   where $1 \leq j \leq m$. Define $F(v) = \sum_{m \geq 1} F_m(x)v^m$ where $F \in \{A, C\}$, $D(v) = \sum_{m \geq 3} D_m(x)v^{m-2}$, $B(v,u) = \sum_{m \geq 1} \sum_{j=1}^m B_{m,j}u^{m-j}v^{m-1}$. Then, by the fact that
\[ C_1(x) = \frac{x}{1-x}, \]  
the above recurrence can be written as

\[
A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v, 1),
\]

\[
B(v, u) = \frac{x(1 + C(v))}{(1 - v)(1 - vu)} + \frac{x}{vu}(B(v, u) - B(v, 0)) + \frac{x}{1-v}B(vu, 1),
\]

\[
C(v) = \frac{x}{(1-x)(1-v)} + \frac{xv}{1-v}D(v),
\]

\[
D(v) = \frac{x}{1-v} + \frac{xv}{1-v}D(v).
\]

Thus, \( D(v) = \frac{x}{1-v-xv} \) and \( C(v) = \frac{x(v^2x^2 + v-1)}{(1-x)(1-v)(vx+v-1)} \). The equation of \( B(v, u) \) with \( u = x/v \) gives

\[
B(v, 0) = \frac{x(v^2x^2 + vx^2 - v^2 + 2v - 1)}{(1-v)(1-x)(vx+v-1)} + \frac{x}{1-v}B(x, 1).
\]

Therefore, by substituting expression of \( B(v, 0) \) into equation of \( B(v, u) \) with \( u = 1 \) and \( v = \frac{1-\sqrt{1-4x}}{2} \), we have

\[
B(x, 1) = \frac{x(x^2\sqrt{1-4x} - x^2 - 2)}{(x^4 - x^3 - 2x^2 + 3x - 1)\sqrt{1-4x} + x^3 - 6x^2 + 5x - 1}.
\]

Hence, by taking \( v = \frac{1-\sqrt{1-4x}}{2} \) and solving for \( A(0) \), we complete the proof.

(2) The rules of the generating trees \( T(\{102, 110, 120, 201\}) \) are given by

\[
a_m \Rightarrow a_{m+1}b_{m,1} \cdots b_{m,m},
\]

\[
b_{m,j} \Rightarrow c_1 \cdots c_j a_{m+2-j} b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \quad 1 \leq j \leq m,
\]

\[
c_m \Rightarrow c_1(0101)^2d_3 \cdots d_m,
\]

\[
d_m \Rightarrow (0101)^2d_3 \cdots d_m,
\]

\[
0101 \Rightarrow 0101,
\]

where \( a_m = 0^m \), \( b_{m,j} = a_{m,j} \) with \( j = 1, 2, \ldots, m \), \( c_m = a_mm(m-1) \), and \( d_m = a_mm(m-1)(m-2) \). Define \( A_m(x) \) (respectively, \( B_{m,j}(x) \), \( C_m(x) \), \( D_m(x) \)) to be the generating function for the number of nodes at level \( n \geq 1 \) for the subtree of \( T(B; a_m) \) (respectively, \( T(B; b_{m,j}) \), \( T(B; c_m) \), \( T(B; d_m) \)), where its root stays at
level 0. Define $B_m(x) = \sum_{j=1}^{m} B_{m,j}(x)$. Thus,

$$A_m(x) = x + xA_{m+1} + xB_m(x),$$

$$B_{m,j}(x) = x + x \sum_{i=1}^{j} C_i(x) + xA_{m+2-j}(x) + xB_{m+1-j}(x),$$

$$C_m(x) = x + \frac{2x^2}{1-x} + xC_1(x) + x \sum_{i=3}^{m} D_i(x),$$

$$D_m(x) = x + \frac{2x^2}{1-x} + x \sum_{i=3}^{m-1} D_i(x),$$

where $1 \leq j \leq m$. Define $F(v) = \sum_{m \geq 1} F_m(x)v^{m-1}$ where $F \in \{A,B,C\}$ and $D(v) = \sum_{m \geq 3} D_m(x)v^{m-3}$. Then, by the fact that $C_1(x) = \frac{x}{(1-x)^2}$, the above recurrence can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v} (A(v) - A(0)) + xB(v),$$

$$B(v) = \frac{x}{(1-v)^2} + \frac{x}{(1-v)^2} C(v) + \frac{x}{v(1-v)} (A(v) - A(0)) + \frac{x}{1-v} B(v),$$

$$C(v) = \frac{x}{(1-x)^2} + \frac{xv}{1-x} \left( 1 + \frac{2x}{1-x} + \frac{x}{(1-x)^2} \right) + \frac{xv^2}{1-v} D(v),$$

$$D(v) = \frac{x}{1-v} + \frac{2x^2}{1-x(1-v)} + \frac{xv^2}{1-v} D(v).$$

It is not hard to get explicit formulas for $D(v), C(v)$. Then by solving the second equation for $B(v)$ and substituting it into first equation, we obtain

$$\frac{(v^2 - v + x)A(v) + (v-1)xA(0)}{vx + v - 1}(1-x)^2(1-v)^2 + \frac{x(1-x)^2v^3(-2x + 3 + (1 + x)v)}{(vx + v - 1)(1-x)^2(1-v)^2} = 0.$$  

Hence, by taking $v = \frac{1 - \sqrt{x(1-x)}}{2}$ and solving for $A(0)$, we complete the proof. \hfill \Box

**Theorem 12 (Class 208).** We have $\{100, 110, 120, 201\} \sim \{100, 110, 120, 210\}$. Moreover, the generating function $F_{\{100,110,120,201\}}(x)$ is given by

$$\frac{x}{1-x} + x \sum_{j \geq 0} \frac{x^{j+1}((1-w_j-x)(2(1-x)w_j^2-x) \Pi_{i=0}^{j-1}(x-w_i))}{1-2x \sum_{j \geq 0} \Pi_{i=0}^{j-1}(x-w_j) \Pi_{j=0}^{i}(1-w_i) \Pi_{i=0}^{j}(1-w_i)},$$

where $w_i = \frac{x^{i+1}}{1-x(1-x)}/(1-x)$ for all $i \geq 0$. 
Proof. By Table 1, we see that the generating tree $T(\{100, 110, 120, 201\})$ is the same as the generating tree $T(\{100, 110, 120, 210\})$. Thus, $\{100, 110, 120, 201\} \sim \{100, 110, 120, 210\}$. Now fix $B = \{100, 110, 120, 201\}$ (for example). Define $A_m(x)$ (respectively, $B_{m,j}(x)$) to be the generating function for the number of nodes at level $n \geq 1$ for the subtree of $T'(B; a_m)$ (respectively, $T'(B; b_{m,j})$), where its root stays at level 0.

Define $A(v) = \sum_{m \geq 1} A_m(x)v^{m-1}$ and $B(v, u) = \sum_{m \geq 1} \sum_{j=1}^{m} B_m(x)v^{m-1}u^{m-j}$. Then, as before, the rules of the generating tree $T(B)$ can be written as

$$A(v) = \frac{x}{1-v} + \frac{x}{v}(A(v) - A(0)) + xB(v, 1), \quad (11)$$

$$B(v, u) = \frac{x}{(1-v)(1-uv)} + \frac{2x}{uv(1-v)}(A(uv) - A(0))$$
$$+ \frac{x}{u(1-v)}(B(v, u) - B(v, 0)) + \frac{x}{1-v}B(uv, 1). \quad (12)$$

By finding $A(v)$ from (11) and substituting it into (12) after replacing $v$ by $v/u$, we obtain

$$B(v/u, u) = \frac{x}{(1-v)^2} + \frac{x}{(1-v)(v-x)} \left( \frac{x}{1-v} - A(0) + xB(v, 1) \right)$$
$$+ \frac{x}{u-v}(B(v/u, u) - B(v/u, 0)) + \frac{x}{u-v}B(v, 1). \quad (13)$$

By taking $u = v + x$, we have

$$B(v/(v+x), 0) = \frac{x}{(1-v)^2} + \frac{2x}{(1-v)(v-x)} \left( \frac{x}{1-v} - A(0) + xB(v, 1) \right)$$
$$+ (v+x)B(v, 1),$$

which, by replacing $v$ by $vx/(1-v)$, implies

$$B(v, 0) = \frac{(2+2x)v^2 - 2(v+x)v + 2A(0) + vx - x}{(1-v)(1-2v)(1-v(1+x))}$$
$$- \frac{xB(vx/(1-v), 1)}{(1-v)(1-2v)}.$$ 

Thus, by setting $u = 1$ into (12) and using expression of $B(v, 0)$, we obtain

$$B(v, 1) = -\frac{(v-x)x^2}{(v^2 + vx - v + x)(1-2v)(1-v)}B(vx/(1-v), 1)$$
$$+ \frac{2x(1-v-x)}{(v^2 + vx - v + x)(1-2v)}A(0)$$
$$+ \frac{x(1-v-x)(2v^2x + 2v^2 - v - x)}{(1-v)(1-2v)(1-v(1+x))(v^2 + vx - v + x)}.$$
By iterating this equation (here we assume that $|x| < 1$), we have

$$B(v, 1) = 2A(0) \sum_{j \geq 0} \frac{x^{2j+1}(1-v_j - x) \prod_{i=0}^{j-1} (x - v_i)}{\prod_{i=0}^{j} (v_i^2 + (x-1)v_i + x) \prod_{i=0}^{j-1} (1 - 2v_i)}$$

$$+ \sum_{j \geq 0} \frac{x^{2j+1}(1-v_j - x)(2(1+x)v_j^2 - v_j - x) \prod_{i=0}^{j-1} (x - v_i)}{(1-v_j(1+x)) \prod_{i=0}^{j} (v_i^2 + (x-1)v_i + x) \prod_{i=0}^{j-1} (1 - 2v_i)}$$

where $v_i = \frac{v_i}{1-v(x-1)/(1-x)}$ for all $i \geq 0$.

By (11) with $v = x$, we have that $A(0) = \frac{x}{1-x} + xB(x, 1)$, which completes the proof. \(\square\)

4. Left to right maximum and combinatorial arguments

In this section, we use bijections based on decompositions of restricted inversion sequences according to left to right maximum to finish the proof for the rest of Theorem 1. In order to do that, we need the following definitions and notation.

A left to right maximum (LRmax) in an inversion sequence $e = e_0e_1\ldots e_n$ is an entry $e_i$ such that $e_i \geq e_j$ for all $j < i$. If $e_i = m$ we say $m$ is the value of the LRmax. Thus for $e = 01102211$, the LRmax are $e_0, e_1, e_2, e_4, e_5$ with values $0, 1, 1, 2, 2$ respectively. So any inversion sequence $e \in I_n$ can be decomposed uniquely as $m_1 \pi_1 \cdots m_k \pi_k$ where $m_1, \ldots, m_k$ are all of the LRmax entries in $e$; thus $0 \leq m_1 \leq \ldots \leq m_k \leq n$ and $m_i > \pi_i$ (entrywise). We call this the WLRmax decomposition of $e$ (W for weakly).

Say $m_1 < m_2 < \cdots < m_k$ are the first occurrences of the (distinct) LRmax values of an inversion sequence $e$. Then $e$ can be decomposed as $m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k$ for some $k \geq 1$ with $m_i \geq \pi_i$ (entrywise) for $1 \leq i \leq k$. This is the LRmax decomposition of $e$. For example, $0003313429$ is an LRmax decomposition with the first occurrence of each left to right maximum value circled (except for the initial $0$).

An inversion sequence $e$ also has a unique decomposition as $e = 0^{r_1} m_2^{r_2} \pi_2 \ldots m_k^{r_k} \pi_k$ with $r_i \geq 1$ for all $i$ where $m_2, \ldots, m_k$ are the first occurrences of the nonzero LRmax values and the first letter of $\pi_i$ (if present) is less than $m_i$ for $i = 2, \ldots, k$. We call this the strict LRmax decomposition of $e$. For example, $00033429939 = 0^3 3^2 \pi_2 4^1 \pi_3 9^2 \pi_4$ with $\pi_2 = 3, \pi_3 = 2, \pi_4 = 39$.

4.1. Class 30: $\{000, 010, 101, 120\} \sim \{000, 010, 110, 120\}$

By the LRmax decomposition of any inversion sequence in either $I_n(\{000, 010, 101, 120\})$ or $I_n(\{000, 010, 110, 120\})$, we have the following lemma.
Lemma 1. Suppose \( e \in I_n \) and \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \) is the LRmax decomposition of \( e \). Then

(1) \( e \in I_n(\{000, 010, 101, 120\}) \) if and only if

(i) If a letter \( m_i \) appears in \( \pi_i \), then it appears as the leftmost letter of \( \pi_i \), for all \( i = 1, 2, \ldots, k; \)

(ii) \( m_1 \pi_1 < m_2 \pi_2 < \cdots < m_k \pi_k; \)

(iii) Each \( \pi_i \) avoids \( \{000, 010, 101, 120\} \).

(2) \( e \in I_n(\{000, 010, 110, 120\}) \) if and only if

(i) If a letter \( m_i \) appears in \( \pi_i \), then it appears as the rightmost letter of \( \pi_i \), for all \( i = 1, 2, \ldots, k; \)

(ii) \( m_1 \pi_1 < m_2 \pi_2 < \cdots < m_k \pi_k; \)

(iii) Each \( \pi_i \) avoids \( \{000, 010, 110, 120\} \).

Now, we are ready to define a recursive bijection

\[
f : I_n(\{000, 010, 101, 120\}) \rightarrow I_n(\{000, 010, 110, 120\}).
\]

We define, \( f(a) = a \), for any letter \( 0 \leq a \leq n \). For any inversion sequence \( e \in I_n(\{000, 010, 101, 120\}) \) with LRmax decomposition \( m_1 \pi_1 \cdot \cdot \cdot m_k \pi_k \), we define \( f(e) = m_1 \beta_1 \cdot \cdot \cdot m_k \beta_k \), where

- if \( \pi_i = m_i \pi'_i \), then \( \beta_i \) is defined to be \( f(\pi'_i)m_i \),
- otherwise, we define \( \beta_i \) as \( f(\pi_i) \).

For example, if \( e = 0022116655487 \), then

\[
f(e) = 0022f(11)66f(554)f(7) = 0022116654587.
\]

By Lemma 1, we have that \( e \in I_n(\{000, 010, 101, 120\}) \) if and only if \( f(e) \in I_n(\{000, 010, 110, 120\}) \).

4.2. Class 36: \( \{000, 010, 100, 201\} \sim \{000, 010, 100, 210\} \)

By the WLRmax decomposition of an inversion sequence in either \( I_n(\{000, 010, 100, 201\}) \) or \( I_n(\{000, 010, 100, 210\}) \), we have the following lemma.

Lemma 2. Suppose \( e \in I_n \) and \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \) is the WLRmax decomposition of \( e \). Then

(1) \( e \in I_n(\{000, 010, 100, 201\}) \) if and only if
(i) Each letter $m_i$ appears at most twice as an LRmax;

(ii) All the letters in $\pi_1 \pi_2 \cdots \pi_k$ are distinct;

(iii) All the letters that are smaller than $m_i$ in $\pi_1 \cdots \pi_k$ form a decreasing sequence.

(2) $e \in I_n(\{000, 010, 100, 210\})$ if and only if

(i) Each letter $m_i$ appears at most twice as an LRmax;

(ii) All the letters in $\pi_1 \pi_2 \cdots \pi_k$ are distinct;

(iii) The letters that are smaller than $m_i$ in $\pi_1 \cdots \pi_k$ form an increasing sequence.

Now, we are ready to define a bijection

$$f : I_n(\{000, 010, 100, 210\}) \rightarrow I_n(\{000, 010, 100, 210\}).$$

Let $e \in I_n(\{000, 010, 100, 210\})$ with WLRmax decomposition $m_1 \pi_1 \cdots m_k \pi_k$. We reorder the letters of $e$ such that $m_1, \ldots, m_k$ stay in their positions, and reorder the letters of $\pi_1 \cdots \pi_k$ such that $\pi_i$ with $i = 1, 2, \ldots, k$ forms an increasing sequence. The result is $f(e)$. For example, if $e = 00212548893$ then $f(e) = 00212538894$. Clearly, $e$ avoids $\{000, 010, 100, 210\}$ if and only if $f(e)$ avoids $\{000, 010, 100, 210\}$.

4.3. Class 37: $\{000, 010, 101, 201\} \sim \{000, 010, 101, 210\}$

Using the strict LRmax decomposition, we have the following characterization of inversion sequences avoiding $\{000, 010, 101, 201\}$ and $\{000, 010, 101, 210\}$ respectively.

**Lemma 3.** Suppose $e \in I_n$ and $0^{r_1} m_2^{a_2} \cdots m_k^{a_k} \pi_k$ is the strict LRmax decomposition of $e$. So, $e \in I_n(\{000, 010, 101, 201\})$ (respectively, $e \in I_n(\{000, 010, 101, 210\})$)

if and only if

(i) each letter in $e$ appears at most twice;

(ii) if a letter $x$ appears twice in $e$ then the two occurrences of $x$ are adjacent;

(iii) the entries in $\pi_1 \cdots \pi_k$ that are less than $m_i$ are weakly decreasing (respectively, weakly increasing) for $i = 1, \ldots, k$.

Here is the bijection. In the strict LRmax decomposition $0^{r_1} m_2^{a_2} \cdots m_k^{a_k} \pi_k$ of $e \in I_n(\{000, 010, 101, 201\})$, write $\pi_2 \cdots \pi_k$ as a list $a_1^s a_2^s \cdots a_k^s$ with superscripts $s_i = 2$ to indicate repeated entries (otherwise $s_i = 1$ and is omitted). For example, for

$$e = 002 \underline{2} 23 \underline{3} 7 \underline{6} 6 \underline{10} 9 8 8 5 1 \in I_{16}(\{000, 010, 101, 201\})$$

with the first occurrence of each nonzero LRmax circled and the $\pi$’s underlined, we have $m_1 = 0, r_1 = 2, m_2 = 2, r_2 = 2, m_3 = 4, m_4 = 7, m_5 = 10$ and
\(\pi_2 = \epsilon, \pi_3 = 3, \pi_4 = 66, \pi_5 = 9988851\) and \(a_1^{s_1}a_2^{s_2}\ldots a_t^{s_t} = 3629^28^251\). In the expression \(a_1^{s_1}a_2^{s_2}\ldots a_t^{s_t}\), arrange \(a_1, \ldots, a_t\) in increasing order while keeping the superscripts frozen in place. The example yields \(13256689\). Replace each “2” superscript with a duplicate entry to get \(133556689\) and split this list into a new set of \(\pi\)'s, say \((\pi'_i)_{i=2}^k\) with \(\pi'_i\) of the same length as \(\pi_i\) for each \(i\). Then the desired inversion sequence in \(I_n(\{000,010,101,210\})\) has strict LRmax decomposition \(0^r m_2^{r_2} \pi'_2 \ldots m_k^{r_k} \pi'_k\). In the example, we get
\[
00(2\ 2\ 3\ 4\ 1\ 7\ 3\ 3(10)\ 5\ 5\ 6\ 6\ 8\ 9) \in I_{16}(\{000,010,101,210\}) \quad (14)
\]
with the first nonzero LRmax entries still circled and each \(\pi'_i\) underlined for clarity.

To reverse the map, list the nonempty \(\pi\)'s as boxed entries with \(m_i\) marked above the sequence of boxes for \(\pi_i\), so that each box is associated with an \(m_i\). The preceding example (14) gives
\[
\begin{array}{cccc}
4 & 7 & 10 \\
1 & 3 & 3 & 5 & 5 & 6 & 6 & 8 & 9
\end{array}
\]
Now form a set \(S\) of the distinct letters in \(\pi_2\ldots\pi_k\), here \(S = \{1,3,5,6,8,9\}\), and erase the contents of the boxes except an “x” is inserted in each box that contained the second occurrence of a repeated letter:
\[
\begin{array}{cccc}
4 & 7 & 10 \\
1 & x & 3 & 3 & 5 & 5 & 6 & 6 & 8 & 9
\end{array}
\]
Next fill in the blank boxes left to right in turn with the letters of \(S\) using the largest available letter that is less than the \(m_i\) associated with the box. The example yields
\[
\begin{array}{cccc}
4 & 7 & 10 \\
3 & x & 6 & x & 9 & x & x & 5 & 1
\end{array}
\]
Lastly, replace each “x” with the letter immediately to its left. This yields the \(\pi_i\) sequence of the original inversion sequence.

4.4. Class 168: \(\{010,100,101,201\}\) \(\sim\) \(\{010,100,101,210\}\)

This case is very similar to Case 37, see Subsection 4.3. Using the strict LRmax decomposition, we have the following characterization of inversion sequences avoiding \(\{010,100,101,201\}\) and \(\{010,100,101,210\}\) respectively.

Lemma 4. Suppose \(e \in I_n\) and \(0^{r_1} m_2^{r_2} \pi_2 \ldots m_k^{r_k} \pi_k\) is the strict LRmax decomposition of \(e\). Then \(e \in I_n(\{010,100,101,201\})\) (respectively, \(e \in I_n(\{010,100,101,210\})\)) if and only if
(i) the letters in \(\pi_2 \ldots \pi_k\) are all distinct;
(ii) no \(m_i\) appears as a letter in \(\pi_2 \ldots \pi_k\);
(iii) the entries in the concatenation \(\pi_i \ldots \pi_k\) that are less than \(m_i\) are decreasing (respectively, increasing) for \(i = 1, \ldots, k\).
Here is the bijection. In the strict LRmax decomposition $0^{r_1}m_2^{r_2}\pi_2\ldots m_k^{r_k}\pi_k$ of $e \in I_n(\{010, 100, 101, 201\})$, rearrange (if necessary) the letters of $\pi_2\ldots\pi_k$ so that they are increasing. For example, with the $\pi_i$’s underlined and the first occurrence of each nonzero LRmax circled,

\[
\begin{align*}
000 \underline{3} \underline{3} 3 2 7 7 6 \underline{5} 4 1 & \in I_{15}(\{010, 100, 101, 201\}) \\
\rightarrow 000 \underline{3} \underline{3} 3 1 7 7 2 \underline{5} 4 6 & \in I_{15}(\{010, 100, 101, 210\}).
\end{align*}
\]

The inverse mapping is obtained much as in Case 37 (but without the complication of repeated entries).

4.5. Class 156: $\{010, 100, 101, 120\} \overset{J}{\sim} \{010, 100, 110, 120\}$

**Lemma 5.** Suppose $e \in I_n$ avoids 100 and $b$ starts a 110 or 101 pattern in $e$. Then $b$ is a left to right maximum in $e$.

**Proof.** If $b$ starts a 110 in $e$ but is not a left to right maximum, then there is a $c > b$ and an $a < b$ such that $cbba$ is a subsequence of $e$. But then $cbb$ is a forbidden 100. Similarly for 101. □

**Lemma 6.** Suppose $e \in I_n(\{010, 100, 120\})$ and $m_1\pi_1 m_2\pi_2 \ldots m_k\pi_k$ is the LRmax decomposition of $e$. Then

(i) $m_1\pi_1 < m_2\pi_2 < \cdots < m_k\pi_k$ (entrywise).

(ii) If $e$ avoids 101, then all occurrences of $m_i$ in $e$ occur as a single run at the start of $m_i\pi_i$, $1 \leq i \leq k$.

(iii) If $e$ avoids 110, then all but the first occurrence of $m_i$ occur as a single (possibly vacuous) run at the end of $\pi_i$, $1 \leq i \leq k$.

**Proof.** (i) Certainly, $m_i\pi_1 < m_i$. If $a \leq m_i$ for some entry $a$ in $\pi_{i+1}$, then $m_i m_{i+1} a$ is a 120 if $a < m_i$ and a 010 if $a = m_i$. Parts (ii) and (iii) are clear. □

In view of Lemmas 5 and 6, the following is a bijection from $I_n(\{010, 100, 101, 120\})$ to $I_n(\{010, 100, 110, 120\})$: in the LRmax decomposition $0^{r_1}m_2^{r_2}\pi_2\ldots m_k^{r_k}\pi_k$ of $e \in I_n(\{010, 100, 101, 120\})$, for each $i = 2, \ldots, k$ move all occurrences of $m_i$ in $\pi_i$ from the start of $\pi_i$ to the end. For example,

\[
\begin{align*}
000 & 3 3 3 2 1 6 4 5 8 8 7 9 10 = \\
000 & \underline{3} \underline{3} 3 2 1 \underline{6} 4 5 \underline{8} 8 7 \underline{9} 10 \rightarrow 000 \underline{3} 2 1 3 3 \underline{6} 4 5 \underline{8} 7 8 \underline{9} 10.
\end{align*}
\]

To reverse the map, move all occurrences of $m_i$ in $\pi_i$ from the end of $\pi_i$ to the start.

4.6. Class 133: $\{000, 101, 120, 201\} \overset{J}{\sim} \{000, 101, 120, 210\}$

We have the following characterizations whose straightforward proofs are left to the reader.
Lemma 7. Suppose \( e \in \mathbf{I}_n \) and \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \) is the LRmax decomposition of \( e \). Then \( e \in \mathbf{I}_n(\{000, 101, 120, 201\}) \) if and only if
(i) \( m_i \leq \pi_{i+1} \) for \( 1 \leq i \leq k-1 \),
(ii) each \( \pi_i \) is weakly decreasing,
(iii) no entry occurs three or more times,
(iv) if \( e_i = m \) is both a LRmax and a descent top, then \( e_i \) is the last occurrence of \( m \) in \( e \).

Lemma 8. Suppose \( e \in \mathbf{I}_n \) and \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \) is the LRmax decomposition of \( e \). Then \( e \in \mathbf{I}_n(\{000, 101, 120, 210\}) \) if and only if
(i) \( m_i \leq \pi_{i+1} \) for \( 1 \leq i \leq k-1 \),
(ii) for each \( i \), if \( \pi_i \) has a descent, then it has only one, it occurs right at the start and the descent top is the second occurrence of \( m_i \),
(iii) no entry occurs three or more times,
(iv) if \( e_i = m \) is both a LRmax and a descent top, then \( e_i \) is the last occurrence of \( m \) in \( e \).

Here is the bijection, with an obvious inverse. Given \( e \in \mathbf{I}_n(\{000, 101, 120, 201\}) \) with LRmax decomposition \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \), for each \( i \geq 2 \), reverse \( \pi_i \) unless the first entry of \( \pi \) is the second occurrence of \( m_i \) in which case leave this entry intact and reverse the rest of \( \pi_i \). For example,

\[
0011254439988766 = 0011254439988766 \rightarrow 001125344966788 .
\]

4.7. Class 164: \( \{010, 100, 120, 201\} \sim \{010, 110, 120, 201\} \)

We have the following characterizations.

Lemma 9. Suppose \( e \in \mathbf{I}_n \) and \( m_1 \pi_1 m_2 \pi_2 \ldots m_k \pi_k \) is the LRmax decomposition of \( e \). Then \( e \in \mathbf{I}_n(\{010, 100, 120, 201\}) \) if and only if
(i) \( m_i < \pi_{i+1} \) for \( 1 \leq i \leq k-1 \), and
(ii) each \( m_i \pi_i \) starts with one or more occurrences of \( m_i \) and thereafter is decreasing except that it may end with zero or more occurrences of \( m_i \).

Proof. If condition (i) is not met, then either \( a = m_i \) occurs in \( \pi_{i+1} \) and \( m_i m_{i+1} a \) is a 010 or \( a < m_i \) occurs in \( \pi_{i+1} \) and \( m_i m_{i+1} a \) is a 120. If \( e \in \mathbf{I}_n(\{010, 100, 120, 201\}) \) and \( m_i = a \) say, then the entries other than \( a \) in \( \pi_i \) are decreasing, for else \( m_i \) starts a 100 or a 201, and all occurrences of \( a \) in \( \pi_i \) are at the start or the end for else \( a \) is the "2" of a 120. Thus condition (ii) holds. We leave the reader to show the converse: that if the two conditions hold, then \( e \in \mathbf{I}_n(\{010, 100, 120, 201\}) \). \[\square\]

The next lemma has an analogous proof.
Lemma 10. Suppose $e \in I_n$ and $m_1\pi_1m_2\pi_2\ldots m_k\pi_k$ is the LRmax decomposition of $e$. Then $e \in I_n((\{010, 110, 120, 201\})$ if and only if

(i) $m_i < \pi_{i+1}$ for $1 \leq i \leq k-1$, and
(ii) each $\pi_i$ has the form $m_i^r w m_i^t$ with $r, t \geq 0$, where $m_i > w$ and $w$ is decreasing except that the last letter of $w$ may be repeated indefinitely.

Here is the bijection, with an obvious inverse. Suppose given $e \in I_n((\{010, 110, 120, 201\})$ with LRmax decomposition $m_1\pi_1m_2\pi_2\ldots m_k\pi_k$. By Lemma 9, for each $i \geq 2$, $\pi_i$ has the form $m_i^r u_1 \ldots u_s m_i^t$ with $m_i > u_1 > \cdots > u_s$ for some $r, s, t \geq 0$. Replace $\pi_i$ with $u_1 \ldots u_s^{r+1} m_i^{s}$ if $s \geq 1$ and leave $\pi$ unchanged if $s = 0$. For example,

$$000332133\overline{7} 765478\overline{10} 10910 \rightarrow 000211133\overline{7} 654478\overline{10} 9910.$$

4.8. Class 166: $\{010, 101, 120, 201\}$

We have the following characterizations whose straightforward proofs are left to the reader.

Lemma 11. Suppose $e \in I_n$ and $m_1\pi_1m_2\pi_2\ldots m_k\pi_k$ is the LRmax decomposition of $e$. Then $e \in I_n((\{010, 101, 120, 201\})$ if and only if

(i) for $1 \leq i \leq k-1$, we have $m_i < \pi_{i+1}$, and
(ii) each $m_i\pi_i$ is weakly decreasing.

Lemma 12. Suppose $e \in I_n$ and $m_1\pi_1m_2\pi_2\ldots m_k\pi_k$ is the LRmax decomposition of $e$. Then $e \in I_n((\{010, 101, 120, 201\})$ if and only if

(i) for $1 \leq i \leq k-1$, we have $m_i < \pi_{i+1}$, and
(ii) for each $i \geq 2$, all occurrences of $b := m_i$ in $\pi_i$ occur at the start and the rest of $\pi_i$ is weakly increasing.

Here is the bijection, with an obvious inverse. For each $\pi_i$, set $b = m_i$, leave all occurrences of $b$ at the start of $\pi_i$ intact, and reverse the remainder of $\pi_i$ to change it from weakly decreasing to weakly increasing. For example, with the first occurrence of each noninitial left-to-right maximum circled

$$0 0 0 3 3 2 2 2 1 6 5 4 4 9 0 3 1 2 2 2 6 4 4 5.$$

4.9. Class 206: $\{101, 100, 120, 201\}$

By the finding the generating trees for all pairs in Class 206, see Table 1, we can state the following result.

Theorem 13 (Class 206). We have
The rules of the generating tree $\mathcal{T}(\{100,101,120,201\})$ are
\[
\begin{align*}
a_m & \leadsto a_{m+1}b_{m,1} \cdots b_{m,m}, \\
b_{m,j} & \leadsto a_{m+1-j}c_{m+2-j,2} \cdots c_{m,j}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \\c_{m,j} & \leadsto a_{m+2-j}c_{m+3-j,2} \cdots c_{m,j-1}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j},
\end{align*}
\]
where $a_m = 0^m$, $b_{m,j} = a_{m,j}$, and $c_{m,j} = a_{m,j}(j - 1)$.

The rules of the generating tree $\mathcal{T}(\{100,101,120,210\})$ are
\[
\begin{align*}
a_m & \leadsto a_{m+1}b_{m,1} \cdots b_{m,m}, \\
b_{m,j} & \leadsto a_{m+1-j}c_{m+2-j,2} \cdots c_{m,j}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \\c_{m,j} & \leadsto a_{m+2-j}c_{m+3-j,2} \cdots c_{m,j-1}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j},
\end{align*}
\]
where $a_m = 0^m$, $b_{m,j} = a_{m,j}$, and $c_{m,j} = a_{m,j}0$.

The rules of the generating tree $\mathcal{T}(\{101,110,120,201\})$ are
\[
\begin{align*}
a_m & \leadsto a_{m+1}b_{m,1} \cdots b_{m,m}, \\
b_{m,j} & \leadsto a_{m+2-j}c_{m+1-j,1} \cdots c_{m,j}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j}, \\c_{m,j} & \leadsto a_{m+2-j,1}c_{m+2-j,1}a_{m+2-j}b_{m+1-j,1} \cdots b_{m+1-j,m+1-j},
\end{align*}
\]
where $a_m = 0^m$, $b_{m,j} = a_{m,j}$, and $c_{m,j} = a_{m,j}(j - 1)$.

We have the following characterizations whose straightforward proofs are left to the reader.

**Lemma 13.** Suppose $e \in I_n$ and $m_1 \pi_1 m_2 \pi_2 \cdots m_k \pi_k$ is the LRmax decomposition of $e$. Then $e \in I_n(\{100,101,120,201\})$ if and only if
(i) for $1 \leq i \leq k - 1$, we have $m_i \leq \pi_{i+1}$ and, further, $m_i < \pi_{i+1}$ unless every entry of $\pi_i$ is equal to $m_i$; and
(ii) each $m_i \pi_i$ starts with one or more occurrences of $m_i$ and thereafter is strictly decreasing.

**Lemma 14.** Suppose $e \in I_n$ and $m_1 \pi_1 m_2 \pi_2 \cdots m_k \pi_k$ is the LRmax decomposition of $e$. Then $e \in I_n(\{101,110,120,201\})$ if and only if
(i) for $1 \leq i \leq k - 1$, we have $m_i \leq \pi_{i+1}$ and, further, $m_i < \pi_{i+1}$ unless every entry of $\pi_i$ is equal to $m_i$; and
(ii) each $m_i \pi_i$ consists of a sequence of one or more letters that is strictly decreasing except that the last letter may be repeated indefinitely.

Here is the bijection, with an obvious inverse. For each $\pi_i$, if all its letters are $m_i$ leave $\pi_i$ intact, otherwise, let $a < m_i$ denote its last letter, set $b = m_i$, and transfer all (if any) occurrences of $b$ at the start of $\pi_i$ to the end, changing them from $b$ to $a$. For example, with the first occurrence of each noninitial left-to-right maximum circled

\[00 \{2 \} 2 2 1 0 \{5 \} 4 3 \{8 \} 8 8 \{9 \} 9 8 \to 00 \{2 \} 1 0 0 0 \{5 \} 4 3 \{8 \} 8 8 \{9 \} 8 8.\]
4.10. Class 207: \( \{100, 101, 110, 210\} \sim \{100, 101, 110, 201\} \)

We have the following characterization of inversion sequences avoiding \( \{100, 101, 110\} \).

**Lemma 15.** Suppose \( e \in I_n \) and \( 0^r m_2^{r_2} \pi_2 \ldots m_k^{r_k} \pi_k \) is the strict LRmax decomposition of \( e \). Then \( e \in I_n(\{100, 101, 110\}) \) if and only if

(i) for \( 2 \leq i \leq k \), either \( m_i \) occurs only once in \( e \) or each occurrence of \( m_i \) is a weak right-to-left min in \( e \) (that is, \( \leq \) all following letters), and

(ii) the concatenation \( \pi_2 \ldots \pi_k \) consists of distinct letters.

**Lemma 16.** Suppose \( e \in I_n(\{100, 101, 110\}) \) and \( 0^r m_2^{r_2} \pi_2 \ldots m_k^{r_k} \pi_k \) is the strict LRmax decomposition of \( e \). Then

(i) \( e \in I_n(\{100, 101, 110, 210\}) \) if and only if the concatenation \( \pi_2 \ldots \pi_k \) is (strictly) increasing, and

(ii) \( e \in I_n(\{100, 101, 110, 201\}) \) if and only if, for \( i = 2, \ldots, k \), the entries in the concatenation \( \pi_2 \ldots \pi_k \) that are less than \( m_i \) form a decreasing list.

Here is the bijection. In the strict LRmax decomposition \( 0^r m_2^{r_2} \pi_2 \ldots m_k^{r_k} \pi_k \) of \( e \in I_n(\{100, 101, 110, 210\}) \), consider the \( \pi \)'s as a set of filled boxes, one entry in each box. Remove the entries from all boxes and refill the boxes left to right, using the largest available entry that is less than \( m_i \) when the box is in \( \pi_i \). For example, with the first occurrence of each left-to-right max circled and the \( \pi \)'s underlined.

\[
0^5 \overset{\circ}{\underline{6}} \underline{0} \underline{1} \underline{7} \underline{8} 2 \underline{3} \underline{12} 4 5 10 15 11 13 14 16 17 \\
\rightarrow 0^5 \overset{\circ}{\underline{6}} \underline{5} \underline{4} \underline{7} \underline{8} 3 \underline{2} \underline{12} 11 10 1 \underline{15} 14 13 0 16 16 \underline{17}.
\]

The inverse is obvious: identify the \( \pi \)'s and then rearrange their entries in increasing order.

**References**


5. Appendix

Table 2: Sequences \( \{I_n(B)\}_{n=0}^\infty \), where \( B \) any set of four patterns in \( P \).

<table>
<thead>
<tr>
<th>Class</th>
<th>( B )</th>
<th>( {I_n(B)}_{n=0}^\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{000,001,010,012}, {000,001,011,012}</td>
<td>1,2,1,0,0,0,0,0,0</td>
</tr>
<tr>
<td>2</td>
<td>{000,001,010,011}, {000,001,011,011}</td>
<td>1,2,1,0,0,0,0,0,0</td>
</tr>
<tr>
<td>3</td>
<td>{000,001,012,110}</td>
<td>1,2,1,0,0,0,0,0,0</td>
</tr>
</tbody>
</table>

References


| Class | $B$ | $(|I_n(B)|)^{+,	ext{st}}$ |
|-------|-----|-----------------|
| 4     | $000.001.012.021$ | $000.001.012.100$ | $000.001.012.101$ |
| 5     | $000.001.012.120$ | $000.010.011.012$ | $1.22.1.0.0.0.0$ |
| 6     | $000.010.010.021$ | $000.001.010.100$ | $000.001.010.101$ |
| 7     | $000.010.010.102$ | $000.001.010.110$ | $000.001.010.120$ |
| 8     | $000.010.010.201$ | $000.001.010.210$ | $000.001.010.220$ |
| 9     | $000.010.011.011$ | $000.001.011.101$ | $000.001.011.120$ |
| 10    | $001.010.011.021$ | $001.010.101.101$ | $001.010.111.120$ |
| 11    | $001.010.102.011$ | $001.010.112.101$ | $001.010.112.210$ |
| 12    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 13    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 14    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 15    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 16    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 17    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 18    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 19    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 20    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 21    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 22    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 23    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 24    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 25    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 26    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 27    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
| 28    | $001.010.112.120$ | $001.010.112.121$ | $001.010.112.210$ |
Continuation of Table 2

<p>| Class | (B) | (\left\lfloor \log_2|B| \right\rfloor + 2) |
|-------|-------|-----------------|
| 39    | 000.010,101,110 | 1.24,10.29,34.02.278 |
| 40    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 41    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 42    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 43    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 44    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 45    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 46    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 47    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 48    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 49    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 50    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 51    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 52    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 53    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 54    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 55    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 56    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 57    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 58    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 59    | 000.010,101,120 | 1.24,10.29,34.02.278 |
| 60    | 000.010,101,120 | 1.24,10.29,34.02.278 |</p>
<table>
<thead>
<tr>
<th>Class</th>
<th>$B$</th>
<th>((1_n(B))_{\infty}^{\text{min}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>[000.010.021.110], [000.010.021.120], [000.010.021.210]</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
<tr>
<td>65</td>
<td>[010.011.100.120], [010.011.110.210]</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
<tr>
<td>66</td>
<td>[010.011.200.110], [010.011.210.120]</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
<tr>
<td>67</td>
<td>[010.011.100.110], [010.011.100.120], [010.011.100.210]</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
<tr>
<td>68</td>
<td>[010.011.200.110], [010.011.210.120]</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
<tr>
<td>69</td>
<td>[012.012.100.110], [012.012.100.120],</td>
<td>1.2 - 4.9 \times 10^{-2}</td>
</tr>
</tbody>
</table>

For classes 10 to 12, the table continues with similar entries, showing the range of values for \((1_n(B))_{\infty}^{\text{min}}\). Each entry includes a range of values followed by a '1,2,5,12' pattern. The table extends to cover various classes with similar entries, indicating a pattern of values across different classes.
| Class | $B$ | $(|L_1(B)|)^2$ |
|-------|-----|----------------|
| 129   | (111,120,201) | 1.254  |
| 130   | (201,010,101) | 1.254  |
| 131   | (111,120,201) | 1.254  |
| 132   | (111,120,201) | 1.254  |
| 133   | (111,120,201) | 1.254  |
| 134   | (111,120,201) | 1.254  |
| 135   | (111,120,201) | 1.254  |
| 136   | (111,120,201) | 1.254  |
| 137   | (111,120,201) | 1.254  |
| 138   | (111,120,201) | 1.254  |
| 139   | (111,120,201) | 1.254  |
| 140   | (111,120,201) | 1.254  |
| 141   | (111,120,201) | 1.254  |
| 142   | (111,120,201) | 1.254  |
| 143   | (111,120,201) | 1.254  |
| 144   | (111,120,201) | 1.254  |
| 145   | (111,120,201) | 1.254  |
| 146   | (111,120,201) | 1.254  |
| 147   | (111,120,201) | 1.254  |
| 148   | (111,120,201) | 1.254  |
| 149   | (111,120,201) | 1.254  |
| 150   | (111,120,201) | 1.254  |
| 151   | (111,120,201) | 1.254  |
| 152   | (111,120,201) | 1.254  |
| 153   | (111,120,201) | 1.254  |
| 154   | (111,120,201) | 1.254  |
| 155   | (111,120,201) | 1.254  |
| 156   | (111,120,201) | 1.254  |
| 157   | (111,120,201) | 1.254  |
| 158   | (111,120,201) | 1.254  |
| 159   | (111,120,201) | 1.254  |
| 160   | (111,120,201) | 1.254  |
| 161   | (111,120,201) | 1.254  |
| 162   | (111,120,201) | 1.254  |
| 163   | (111,120,201) | 1.254  |
| 164   | (111,120,201) | 1.254  |
| 165   | (111,120,201) | 1.254  |
| 166   | (111,120,201) | 1.254  |
| 167   | (111,120,201) | 1.254  |
| 168   | (111,120,201) | 1.254  |
| 169   | (111,120,201) | 1.254  |
| 170   | (111,120,201) | 1.254  |
| 171   | (111,120,201) | 1.254  |
| 172   | (111,120,201) | 1.254  |
| 173   | (111,120,201) | 1.254  |
| 174   | (201,010,101) | 1.254  |
| 175   | (201,010,101) | 1.254  |
| 176   | (201,010,101) | 1.254  |
| 177   | (201,010,101) | 1.254  |
| 178   | (201,010,101) | 1.254  |
| 179   | (201,010,101) | 1.254  |
| 180   | (201,010,101) | 1.254  |
| 181   | (201,010,101) | 1.254  |
| 182   | (201,010,101) | 1.254  |
| 183   | (201,010,101) | 1.254  |
| 184   | (201,010,101) | 1.254  |
| 185   | (201,010,101) | 1.254  |
| 186   | (201,010,101) | 1.254  |
| 187   | (201,010,101) | 1.254  |
| 188   | (201,010,101) | 1.254  |
| 189   | (201,010,101) | 1.254  |
| 190   | (201,010,101) | 1.254  |
### Continuation of Table 2

<table>
<thead>
<tr>
<th>Class</th>
<th>$B$</th>
<th>$([H_n(B)])_{n=0}^{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1194</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1195</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1196</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1197</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1198</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1199</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1200</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1201</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1202</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1203</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1204</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1205</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1206</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1207</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1208</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1209</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
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<td>1210</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
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<td>1211</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
<tr>
<td>1212</td>
<td>${01,01,10,21}, {02,11,10,21}$</td>
<td>1.26, 20.60, 9.24, 3.98, 1.92</td>
</tr>
</tbody>
</table>

**End of Table 2**