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STOCHASTIC REPRESENTATIONS OF THE MATRIX VARIATE SKEW ELLIPTICALLY CONTOURED DISTRIBUTIONS

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Abstract

Matrix variate skew elliptically contoured distributions generalize several classes of important distributions. This paper defines and explores matrix variate skew elliptically contoured distributions. In particular, we discuss two stochastic representations of the matrix variate skew elliptically contoured distributions.

1. Introduction

Although a collection of random variables can always be arranged into a vector or matrix, the generalization of univariate distribution results to analogous multivariate results is not necessarily obvious or immediate. Certainly, there are some properties whose extension to a matrix or multivariate context is straightforward, but just as often such extensions can be extremely difficult. For example, if X is a normally-distributed univariate random variable, i.e., $X \sim N(0, 1)$, then the expectation $\mathbb{E}$ of X satisfies

$$\mathbb{E}\{\Phi(hX + k)\} = \Phi(k/\sqrt{1+h^2})$$

for any real scalars h and k. However, it is not obvious how this univariate normal distribution property could be generalized to the context of multivariate normal distributions, and further, to that of matrix variate normal distributions (see [1] for details). Likewise, the matrix variate generalization of the power exponential family of distributions is far from obvious [8].

However, obtaining properties of matrix and multivariate skew distributions is important in both theory and applications. Wen and Zhu derived stochastic representations and the first two moments, among other properties, of the multivariate skew Pearson type VII distribution and the skew t-distribution [9, 10]. In addition, Chen and Gupta extended several important properties of the multivariate skew normal distributions to the matrix variate case [1]. Furthermore, Harrar and Gupta discussed more general matrix variate skew normal distributions [7]. In particular, they obtained the stochastic representation of a subfamily of matrix variate skew normal distributions.

Elliptically-contoured distributions are also an important class of multivariate distributions closely related to several different families of multivariate distributions. This paper, motivated by and in parallel with [7], derives stochastic representations of the matrix variate skew elliptically contoured distributions (MSE).

In particular, we extend several results in [2-4], and the multivariate skew Pearson type VII distributions and skew *t*-distributions in Wen and Zhu [9, 10] to the matrix variate case.

In Section 2, we recall several definitions pertinent to the paper. In Section 3, we obtain stochastic representations of the matrix variate skew elliptically contoured distributions (MSE).

2. Notation and Definitions

Matrix variate distributions have been studied by Gupta and Nagar among others [5]. We recall the definitions and notation they introduced. In particular, we use their definition of a matrix variate skew normal distribution.

Let X be a $p \times m$ random matrix, which has a matrix variate normal distribution, i.e., $X \sim N_{p,m}(M, \Sigma \otimes \Psi)$, where $M \in \mathbb{R}^{p\times m}$ is a $p \times m$ mean matrix, Σ is a $p \times p$ positive definite matrix and Ψ is an $m \times m$ positive definite matrix. We use bold face upper and lower case letters to denote vectors, upper case letters without bold face for matrices, and lower case letters without bold face for elements of a vector or a matrix.

Definition 1. The density of a normal matrix variate X takes the following form:

$$f(X) = (2\pi)^{-pm/2} |\Sigma|^{-m/2} |\Psi|^{-p/2} \operatorname{etr}\left[-\frac{1}{2}\Sigma^{-1}(X - M)\Psi^{-1}(X - M)'\right] \tag{1}$$

which is denoted by $\phi_{p,m}(X, M, \Sigma \otimes \Psi)$, or $\phi_{p,m}(X, \Sigma \otimes \Psi)$ if the mean matrix M is null, where for A a square matrix, $\operatorname{etr}(A) = \exp\{trace(A)\}$.

In addition, the cdf of an m-dimensional normal random vector with mean vector **M**, covariance matrix Ψ, taking value at vector **b**, is denoted by $\Phi_m(\mathbf{b}, \mathbf{M}, \Psi)$, or $\Phi_m(\mathbf{b}, \Psi)$, if the mean vector **M** is null.

Definition 2. The density of a matrix variate Pearson type VII X takes the following form:

$$f(X) = \frac{\Gamma(q)}{(\pi\lambda)^{pm/2}\Gamma(q - pm/2)|\Sigma|^{m/2}|\Psi|^{p/2}}$$

$$\times\left(1 + \frac{\mathrm{tr}((X - M)'\Sigma^{-1}(X - M)\Psi^{-1})}{\lambda}\right)^{-q}, \tag{2}$$

where $q, \lambda > 0$, $q > \frac{pm}{2}$.

We denote the matrix variate Pearson type VII distribution defined above by $P\mathrm{VII}_{p,m}(M, \Sigma, \Psi, q, r)$. Particularly, when $q = \frac{pm + \lambda}{2}$, X is said to have a matrix variate t-distribution with r degrees of freedom and it is denoted by $MT_{p,m}(\lambda, M, \Sigma \otimes \Psi)$.

When $\lambda = 1$, (2) reduces to a matrix variate Cauchy distribution, denoted by $MC_{p,m}(1, M, \Sigma \otimes \Psi)$. When $p = 1$, the *matrix* variate Pearson type VII distribution reduces to the *multivariate* Pearson type VII distribution, denoted by $P\mathrm{VII}_m(\mathbf{M}, \Psi, q, \lambda)$, whose *pdf* is given by

$$f(\mathbf{X}) = \frac{\Gamma(q)}{(\pi\lambda)^{m/2}\Gamma(q - m/2)|\Psi|^{1/2}}\left(1 + \frac{(\mathbf{X} - \mathbf{M})\Psi^{-1}(\mathbf{X} - \mathbf{M})'}{\lambda}\right)^{-q}, \tag{3}$$

where $q > \frac{m}{2}$. Particularly, when $q = \frac{m + \lambda}{2}$, X is said to have a multivariate t-distribution with r degrees of freedom and it is denoted by $MT_m(\lambda, \mathbf{M}, \Psi)$. When $\lambda = p = 1$, Definition 2 reduces to multivariate Cauchy distribution, denoted by $MC_m(M, \Psi)$.

Definition 3. Suppose that $X\ (p \times m)$ is a random matrix. Then X is said to have a *matrix variate elliptically contoured* (*MEC*) *distribution* if it has a characteristic function $\phi_X(T) = \text{etr}(iT'X)\psi(\text{tr}(T'\Sigma T\Psi))$, where $\psi : [0, \infty) \to \mathbb{R}$. This distribution is denoted by $E_{p,m}(M, \Sigma \otimes \Psi, \psi)$.

An MEC reduces to a multivariate elliptically contoured distribution if $p = 1$, denoted by $E_m(\mathbf{M}, \Psi, \psi)$, where $\mathbf{M}$ is an m-dimensional mean vector. By Theorem 2.2.1 in [6], $X \sim E_{p,m}(M, \Sigma \otimes \Psi, \psi)$ if and only if the probability density function (*pdf*) of X is given by

$$f(X) = |\Sigma|^{-m/2}|\Psi|^{-p/2}h(\text{tr}((X - M)'\Sigma^{-1}(X - M)\Psi^{-1})), \tag{4}$$

where h and ψ determine each other for specified p and m.

One can see that the matrix variate normal distributions and matrix variate Pearson type VII distributions are special cases of MEC distributions. If X is distributed as matrix variate normal $N_{p,m}(M, \Sigma \otimes \Psi)$, then $\psi(x) = \exp(-x/2)$ and the characteristic function of X can be written as

$$\phi_X(T) = \text{etr}(iT'X)\text{etr}\left(-\frac{1}{2}T'\Sigma T\Psi\right).$$

Also, $h(x) = (2\pi)^{-pm/2}e^{-x/2}$. If X is distributed as matrix variate Pearson type VII, i.e., $X \sim PVII_{p,m}(M, \Sigma, \Psi, q, \lambda)$, then $\psi(x) = (1 + x/\lambda)^{-q}$, the characteristic function of X can be written as

$$\phi_X(T) = \text{etr}(iT'X)\left(1 + \frac{tr(T'\Sigma T\Psi)}{\lambda}\right)^{-q} \tag{5}$$

and the function h can be written as

$$h(x) = \frac{\Gamma(q)}{(\pi\lambda)^{pm/2}\Gamma\left(q - \frac{pm}{2}\right)}\left(1 + \frac{x}{\lambda}\right)^{-q}. \tag{6}$$

Definition 4. Assume $Z((p+1)\times m) = \begin{pmatrix} \mathbf{X}_0 \\ X \end{pmatrix} \sim E_{p+1,m}(0, \Sigma^* \otimes \Psi, \psi)$, where $\Sigma^* = \begin{pmatrix} 1 & \mathbf{k} \\ \mathbf{k}' & \Sigma \end{pmatrix}$, Σ is a $p \times p$ positive definite matrix, Ψ is an $m \times m$ positive definite matrix, $\mathbf{X}_0$ is $1 \times m$, X is $p \times m$. Then the random matrix $X_{p\times m}$ is said to have a *matrix variate skew elliptically contoured distribution* with parameters $\mathbf{b}, \Sigma, \Psi, \psi$, denoted by $X \sim MSE_{p,m}(\mathbf{b}, \Sigma, \Psi, \psi)$ if the pdf of X has the form

$$f(X) = [F(0, \Psi, \psi)]^{-1} f(X; \Sigma, \Psi, \psi) F(X'\mathbf{b}, \Psi, \psi_{q(X)}), \tag{7}$$

where $\mathbf{b} = \Sigma^{-1}\mathbf{k}'(1-\mathbf{k}\Sigma\mathbf{k}')^{-\frac{1}{2}}$, $q(X) = \mathrm{tr}(X'\Sigma^{-1}X\Psi^{-1})$, $F(\cdot; \Psi, \psi)$ is the cumulative distribution function (cdf) of $E_m(0, \Psi, \psi)$, $f(\cdot; \Psi, \Sigma, \psi)$ is the pdf of $E_{p,m}(0, \Sigma \otimes \Psi, \psi)$, $F(\cdot; \Psi, \psi_{q(X)})$ is the cdf of $E_m(0, \Psi, \psi_{q(X)})$.

Definition 5. The random matrix X $(p \times m)$ is said to have a *matrix variate skew normal distribution*, written as $X \sim MSN_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega)$, if its pdf is given by

$$[\Phi_m(0, \Omega + \mathbf{b}'\Sigma\mathbf{b}\Psi)]^{-1}\phi_{p,m}(X, \Sigma \otimes \Psi)\Phi_m(X'\mathbf{b}, \Omega), \tag{8}$$

where $\mathbf{b} \in R^m$ is a vector of shape parameters, other notation are the same as defined in Definition 1.

Besides the above stochastic representation of the matrix variate skew elliptical contoured distribution, we introduce a different one, the linear transformation method. The first theorem in Section 3 provides an alternative stochastic representation for a subfamily of $MSE_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega, \psi)$.

3. Stochastic Representations of MSE

Multivariate skew normal distributions have been shown to arise from multivariate normal distributions by truncating on some of the variates. This property can be generalized to the matrix variate skew elliptically contoured distributions. The following theorem provides a stochastic representation for the matrix variate skew elliptically contoured distributions of the type $MSE_{p,m}(\mathbf{b}, \Sigma, \Psi, \Omega, \psi)$.

Theorem 1. *Assume that* $Z((p+1)\times m)$ *as in Definition* 4. *Then the conditional pdf of X given the constraint* $\mathbf{X}_0 > 0$ *is* $MSE_{p,m}(\mathbf{b}, \Sigma, \Psi, \psi)$ *as defined in* (7).

Proof. It follows from Theorem 2.3.1 in [6] that the marginal distribution of $\mathbf{X}_0$ is $E_m(0, \Psi, \psi)$, which we denote by $f(\mathbf{X}_0)$, and the marginal distribution of X is $E_{p,m}(0, \Sigma\otimes\Psi, \psi)$, which we denote by $f(X)$. On the other hand, it follows from Theorem 2.6.4 in [6] that the conditional distribution of $\mathbf{X}_0$ given X is $E_m(X'\Sigma^{-1}\mathbf{k}', (1-\mathbf{k}\Sigma\mathbf{k}')\Psi, \psi_{q(X)})$, where $q(X) = \mathrm{tr}(X'\Sigma^{-1}X\Psi^{-1})$. We denote the density of this distribution by $f(\mathbf{X}_0 \mid X)$. Therefore, the *pdf* of X under the constraint $\mathbf{X}_0 > 0$ can be written as

$$f(X \mid \mathbf{X}_0 > 0) = \left[\int_{\mathbf{X}_0<0} f(\mathbf{X}_0)d\mathbf{X}_0\right]^{-1} \times f(X) \times \left[\int_{\mathbf{X}_0>0} f(\mathbf{X}_0 \mid X)d\mathbf{X}_0\right]$$

$$= [F(0, \Psi, \psi)]^{-1} f(X, \Sigma\otimes\Psi, \psi)F(\mathbf{b}, \Psi, \psi_{q(X)}). \tag{9}$$

□

Theorem 1 immediately leads to the following two corollaries.

Corollary 1. *Assume that* $Z((p+1)\times m) = \begin{pmatrix}\mathbf{X}_0 \\ X\end{pmatrix}$ *has a matrix variate Pearson type VII distribution, i.e.,* $Z \sim PVII_{p+1,m}(0, \Sigma^*, \Psi, q, \lambda)$. *Then the distribution of X under the constraint* $\mathbf{X}_0 > 0$ *is called a matrix variate skew Pearson type VII distribution, denoted by* $MSPVII_{p,m}(\mathbf{b}, \Sigma, \Psi, q, \lambda)$, *whose pdf can be written as*

$$[F(0, \Psi, q, r)]^{-1} f(X, \Sigma, \Psi, q, \lambda)F(\mathbf{b}, \Psi, q, \lambda),$$

where $F(.; \Psi, q, r)$ *is the cdf of* $PVII_m(0, \Psi, q, \lambda)$, $f(.; \Sigma, \Psi, \psi, q, \lambda)$ *is the pdf of* $PVII_{p,m}(0, \Sigma, \Psi, q, r)$.

Corollary 2. *Assume that* $Z((p+1)\times m) = \begin{pmatrix} \mathbf{X}_0 \\ X \end{pmatrix}$ *has a matrix variate normal distribution,* $Z \sim N_{p+1,m}(0, \Sigma^* \otimes \Psi)$. *Then the distribution of* X *under the constraint* $\mathbf{X}_0 > 0$ *is called a matrix variate skew normal, denoted by* $MSN_{p,m}(\mathbf{b}, 0, \Sigma, \Psi, \Omega)$, *whose pdf can be written as*

$$[\Phi_m(0, 0, \Psi)]^{-1}\phi_{p,m}(X, 0, \Sigma\otimes\Psi)\Phi_m(X'\mathbf{b}, 0, \Omega),$$

where $\mathbf{b} = \Sigma^{-1}\mathbf{k}'$ *and* $\Omega = (1 - \mathbf{k}\Sigma\mathbf{k}')\Psi$.

A more general matrix variate skew normal distribution was defined by Harrar and Gupta [7]. Moreover, we also have the following theorem.

Theorem 2. *Assume that* $V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim E_{p+1,m}(0, \Sigma^* \otimes \Psi, \psi)$, $\Sigma^* = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix}$, Σ *is* $p\times p$, Ψ *is* $m \times m$, $\mathbf{V}_0$ *is* $1\times m$, V_1 *is* $p\times m$. *For* $\delta_j \in (-1, 1)$, *let*

$$\begin{cases} \mathbf{X}_0 = |\mathbf{V}_0|, \quad \mathbf{X}_j = \delta_j|\mathbf{V}_0| + (1-\delta_j^2)^{1/2}\mathbf{V}_j, \quad j = 1, 2, \ldots, p, \\ |\mathbf{V}_0| = (|v_{01}|, |v_{02}|, \ldots, |v_{0m}|), \\ Z = (\mathbf{X}_0', X')', \quad X = (\mathbf{X}_1', \ldots, \mathbf{X}_p')', \\ V^* = \begin{pmatrix} |\mathbf{V}_0| \\ V_1 \end{pmatrix}, \\ p_c = \int_{\mathbf{X}_0<0} |\Psi|^{-1/2} h(\mathrm{tr}(\mathbf{X}_0\Psi^{-1}\mathbf{X}_0'))d\mathbf{X}_0. \end{cases}$$

Then the random matrix X has a matrix variate skew elliptically contoured distribution. Specifically,

$$X \sim MSE_{p,m}(\Sigma_\delta^{-1}\delta,\ \Sigma_\delta,\ \Psi,\ \Omega_{11.2}\Psi,\ \psi),$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_p)'$, $\Sigma_\delta = \delta\delta' + \Delta\Sigma\Delta$,

$$\Delta = \operatorname{diag}\left((1-\delta_1^2)^{\frac{1}{2}}, \ldots, (1-\delta_p^2)^{\frac{1}{2}}\right)$$

and

$$\Omega_{11.2} = 1 - \delta'\Sigma_\delta^{-1}\delta.$$

Proof. The density of V^* is $p_c^{-1}F(0, 0, \Sigma^* \otimes \Psi, \psi)$, where $F(\cdot, 0, \Sigma^* \otimes \Psi, \psi)$ is the cdf of $E_{p+1,m}(0, \Sigma^* \otimes \Psi, \psi)$. One can see that the pdf of X can be written as

$$f(X) = p_c^{-1}\int_{\mathbf{X}_0 \geq 0} |\,\Sigma^*\,|^{-\frac{1}{2}} |\,\Psi\,|^{\frac{p+1}{2}} h(\operatorname{tr}(Z'\Sigma^{*-1}Z\Psi^{-1}))\,d\mathbf{X}_0$$

$$= p_c^{-1} f_1(X)\int_{\mathbf{X}_0 \geq 0} f(\mathbf{X}_0 \,|\, X)\,d\mathbf{X}_0,$$

where

$$f_1(X) = |\,\Sigma_\delta\,|^{-m/2} |\,\Psi\,|^{-p/2} h(\operatorname{tr}(X'\Sigma_\delta^{-1}X\Psi^{-1})).$$

The distribution of Z is

$$p_c^{-1} |\,J\,|\,|\,\Sigma^*\,|^{-\frac{m}{2}} |\,\Psi\,|^{-\frac{p+1}{2}} h(\operatorname{tr}(\mathbf{X}_\delta'\Sigma^{*-1}\mathbf{X}_\delta\Psi^{-1})),$$

where

$$\mathbf{X}_\delta' = (\mathbf{X}_0', (1-\delta_1^2)^{-\frac{1}{2}}\mathbf{X}_1' - \beta_1\mathbf{X}_0', \ldots, (1-\delta_p^2)^{-\frac{1}{2}}\mathbf{X}_p' - \beta_p\mathbf{X}_0').$$

Let $B = \begin{pmatrix} 1 & 0 \\ \delta & \Delta \end{pmatrix}$. Then

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{-\delta_1}{(1-\delta_1^2)^{\frac{1}{2}}} & (1-\delta_1^2)^{-\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{-\delta_p}{(1-\delta_p^2)^{\frac{1}{2}}} & 0 & 0 & \cdots & (1-\delta_p^2)^{-\frac{1}{2}} \end{pmatrix}.$$

Hence, $Z = BV^*$ and $J(V^* \to Z) = (|B|^{-1})^m = [\prod_{i=1}^{p}(1-\delta_i^2)^{1/2}]^{-m}$.
Therefore, the density of Z can be written as

$$f(Z) = p_c^{-1}|J||\Sigma^*|^{-\frac{m}{2}}|\Psi|^{-\frac{p+1}{2}}h(\mathrm{tr}(B^{-1}Z)'\Sigma^{*-1}(B^{-1}Z)\Psi^{-1})$$

$$= p_c^{-1}|J||\Sigma^*|^{-\frac{m}{2}}|\Psi|^{-\frac{p+1}{2}}h(\mathrm{tr}(Z'\Omega^{-1}Z\Psi^{-1})),$$

where $\Omega = B\Sigma^* B' = \begin{pmatrix} 1 & \delta' \\ \delta & \Sigma_\delta \end{pmatrix} \triangleq \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$.

Consequently, we have

$$f(Z) = p_c^{-1}|\Omega|^{-m/2}|\Psi|^{-(p+1)/2}h(\mathrm{tr}(Z'\Omega^{-1}Z\Psi^{-1})),$$

since $|J||\Sigma^*|^{-n/2} = |B^{-m}||\Sigma^*|^{-m/2} = |\Omega|^{-m/2}$.

Therefore, based on Theorem 2.6.4 in [6], the pdf of X can be written as

$$f(X) = p_c^{-1}|\Omega|^{-m/2}|\Psi|^{-(p+1)/2}\int_{\mathbf{X}_0 \geq 0} h(\mathrm{tr}(Z'\Omega^{-1}Z\Psi^{-1}))d\mathbf{X}_0 \qquad (10)$$

$$= p_c^{-1}f_1(X)\int_{\mathbf{X}_0 > 0} f(\mathbf{X}_0 | X)d\mathbf{X}_0, \qquad (11)$$

where

$$f(\mathbf{X}_0 \mid X)$$

$$= |\,\Omega_{11.2}\Psi\,|^{-1/2} h\big(\mathrm{tr}((\mathbf{X}_0 - \delta' \Sigma_\delta^{-1} X)'(\mathbf{X}_0 - \delta' \Sigma_\delta^{-1} X)(\Omega_{11.2}\Psi)^{-1})\big)$$

and $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21} = 1 - \delta' \Sigma_\delta^{-1}\delta.$ □

In the following corollaries, we use the notation and definitions found in Theorem 2.

Corollary 3. *Assume that* $V = \begin{pmatrix} \mathbf{V_0} \\ V_1 \end{pmatrix} \sim N_{p+1,\,m}(0,\, \Sigma^* \otimes \Psi)$. *Then the random matrix X has a matrix variate skew normal distribution. The pdf of the random matrix X can be written as*

$$f(X) = \left[\int_{\mathbf{X}_0 < 0} \det(\Psi)^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}m} \exp\left\{-\frac{1}{2}\mathbf{X}_0 \Psi^{-1} \mathbf{X}_0'\right\} d\mathbf{X}_0\right]^{-1}$$

$$\times \det(\delta\delta' + \Delta\Sigma\Delta)^{-\frac{1}{2}m} \det(\Psi)^{-\frac{1}{2}p} (2\pi)^{-\frac{1}{2}pm}$$

$$\times \exp\left\{-\frac{1}{2}\mathrm{tr}(X' \Sigma_\delta^{-1} X \Psi^{-1})\right\}$$

$$\times \int_{\mathbf{X}_0 < 0} \det(\Omega_{11.2})^{-\frac{1}{2}m} \det(\Psi)^{-\frac{1}{2}} (2\pi)^{-\frac{1}{2}m}$$

$$\times \exp\left\{-\frac{1}{2}\mathrm{tr}((\mathbf{X}_0 - \delta' \Sigma_\delta^{-1} X')' \Omega_{11.2}^{-1} (\mathbf{X}_0 - \delta' \Sigma_\delta^{-1} X) \Psi^{-1})\right\} d\mathbf{X}_0$$

$$= \left[\int_{\mathbf{X}_0 < 0} \phi_m(\mathbf{X}_0,\, 0,\, \Psi)\, d\mathbf{X}_0\right]^{-1} \phi_{p,\,m}(X,\, 0,\, \Sigma_\delta \otimes \Psi)$$

$$\times \Phi_m(X' \Sigma_\delta^{-1}\delta,\, 0,\, \Omega_{11.2}\Psi)$$

which is denoted by $MSN_{p,\,m}(\Sigma_\delta^{-1}\delta,\, \Sigma_\delta,\, \Psi,\, \Omega_{11.2}\Psi)$.

Proof. First, we define the function h in Theorem 2. We let $h(x) = (2\pi)^{-pm/2} \times e^{-x/2}$. Then Corollary 2 can be proved using (4) in Definition 3 and Theorem 2. □

Corollary 4. *Assume that* $\begin{pmatrix} \mathbf{V_0} \\ V_1 \end{pmatrix} \sim PVII_{p+1,m}(0, \Sigma^*, \Psi, q, r)$. *Then the random matrix X has a matrix variate skew Pearson type VII distribution. The pdf of the random matrix X can be written as*

$$f(X) = \left[\int_{\mathbf{X}_0<0} \frac{\Gamma(q)}{(\pi\lambda)^{m/2}\Gamma(q-m/2)|\Psi|^{1/2}}\left(1+\frac{\mathbf{X}_0\Psi^{-1}\mathbf{X}_0'}{\lambda}\right)^{-q} d\mathbf{X}_0\right]^{-1}$$

$$\times \frac{\Gamma(q)}{(\pi\lambda)^{pm/2}\Gamma(q-pm/2)|\Sigma_\delta|^{m/2}|\Psi|^{p/2}}\left(1+\frac{\mathrm{tr}(X'\Sigma_\delta^{-1}X\Psi^{-1})}{\lambda}\right)^{-q}$$

$$\times \int_{\mathbf{X}_0<0} \frac{\Gamma(q)}{(\pi\lambda)^{m/2}\Gamma(q-m/2)|\Omega_{11.2}\Psi|^{1/2}}$$

$$\times \left\{1+\frac{[\mathbf{X}_0-\delta'(\Sigma_\delta)^{-1}X](\Omega_{11.2}\Psi)^{-1}[\mathbf{X}_0-\delta'(\Sigma_\delta)^{-1}X]'}{\lambda}\right\}^{-q} d\mathbf{X}_0$$

$$= [F(0, 0, \Psi, q, \lambda)]^{-1} f[X, \Sigma_\delta \otimes \Psi, q, \lambda]$$

$$\times F[X'\Sigma_\delta^{-1}\delta, 0, \Omega_{11.2}\Psi, q, \lambda],$$

where $F[\cdot, 0, \Psi, q, \lambda]$ *is the corresponding cdf to the pdf defined in* (3), $f[\cdot, \Sigma_\delta \otimes \Psi, q, \lambda]$ *is the pdf defined in* (2) *when* $M = 0$. *This matrix variate skew Pearson type VII distribution is denoted by*

$$MSPVII_{p,m}(\Sigma_\delta^{-1}\delta, \Sigma_\delta, \Psi, \Omega_{11.2}\Psi, q, \lambda).$$

Proof. First, we define the function h in Theorem 2 as (6). Then Corollary 3 can be proved using (4) in Definition 3 and Theorem 2. □

In the next theorem, we use the same notation and definitions as in Theorem 2.

Theorem 3. *Assume that* $V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim E_{p+1, m}(0, \Sigma^* \otimes \Psi, \psi)$. *Also, assume that* $\Sigma = AA'$, *and* $\Psi = BB'$. *For* $\delta_j \in (-1, 1)$, *let*

$$\begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_p \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} |\, \mathbf{V}_0 \,| + \Delta \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_p \end{pmatrix} \text{ or } X = \delta |\, \mathbf{V}_0 \,| + \Delta V_1.$$

Then the random matrix X has the following stochastic representation:

$$X = R(\delta |\, \mathbf{U}_1 \,| + \Delta A U_2) B', \tag{12}$$

where R *is a nonnegative random variable,* $\mathbf{U}_1$ *is* $1 \times m$ *and uniformly distributed on sphere* S_m, U_2 *is* $p \times m$ *and* $vec(U_2')$ *is uniformly distributed on sphere* S_{pm}. *In addition,* $R, \mathbf{U}_1$ *and* U_2 *are independent. Furthermore,* $\psi(x) = \int_0^\infty \Omega_{pm}(r^2 x) dF(r)$, $x \geq 0$, *where* $\Omega_{pm}(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{pm}$ *denotes the characteristic function of* $vec(U_2')$, *and* $F(r)$ *denotes the distribution function of* r.

Proof. By Theorem 2.5.2 in [6], the stochastic representation for the random matrix V is

$$\begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} RUB',$$

where R is a nonnegative random variable, depending on function ψ. Also, U is $(p+1) \times m$, $vec(U')$ is uniformly distributed on sphere $S_{(p+1)m}$, and R and U are independent. The marginal distribution $\mathbf{V}_0$ of V has the stochastic representation $\mathbf{V}_0 = R\mathbf{U}_1 B'$. Thus, $\delta |\, \mathbf{V}_0 \,|$ has the stochastic representation $\delta |\, \mathbf{V}_0 \,| = R\delta |\, \mathbf{U}_1 \,| B'$. Furthermore, the marginal distribution

V_1 of V has the stochastic representation $V_1 = RAU_2B'$. Therefore, the random matrix X has the stochastic representation given in (12). □

The next corollary follows immediately. Moreover, its statement uses the same notation as in Theorem 2 and also assumes that $\sum = AA'$, $\Psi = BB'$.

Corollary 5. *Assume that* $V = \begin{pmatrix} \mathbf{V}_0 \\ V_1 \end{pmatrix} \sim N_{p+1, m}(0, \sum^* \otimes \Psi)$. *For* $\delta_j \in (-1, 1)$, *let*

$$\begin{pmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_p \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_p \end{pmatrix} |\, \mathbf{V}_0 \,| + \Delta \begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_p \end{pmatrix} \text{ or } Z = \delta |\, \mathbf{V}_0 \,| + \Delta V_1.$$

Then the random matrix Z is distributed as matrix variate skew normal and has the following stochastic representation:

$$Z = R_0(\delta |\, \mathbf{U}_1 \,| + \Delta A U_2) B', \tag{13}$$

where $\mathbf{U}_1, U_2, A, B$ *are the same as in Theorem* 3.

However, it should be noted that R_0 is not necessarily the same as R in Theorem 3. The random variables R and R_0 depend on the characteristic function $\phi(\cdot)$ or the function $h(\cdot)$, or the function $\psi(\cdot)$ of the corresponding distribution, discussed in Definition 3.

4. Concluding Remarks

The set of skew elliptically contoured distributions contains the elliptically contoured distributions. They also have some important properties analogous to the elliptically contoured distributions. They are useful in studying robustness as well as in applications.

In this paper, we defined and explored matrix variate skew elliptically contoured distribution, obtaining in the process two stochastic representations. In addition, we analyzed two subclasses - the matrix variate skew Pearson type VII distributions and the matrix variate skew normal distributions, along with the relationships between them.

We hope to follow these results up rather immediately with some results concerning the moments of a subfamily of this large family of distributions. Future directions include the extension of these results to a general class of the matrix variate elliptically contoured distributions. The quadratic forms of random matrices with elliptically contoured distributions and their characteristic functions are some other possible research directions.

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