

## ESTIMATION OF THE ERROR AUTOCORRELATION MATRIX IN SEMIPARAMETRIC MODEL FOR fMRI DATA

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*Abstract:* In statistical analysis of functional magnetic resonance imaging (fMRI), dealing with the temporal correlation is a major challenge in assessing changes within voxels. This paper aims to address this issue by considering a semiparametric model for single-voxel fMRI. For the error process in the semiparametric model with autocorrelation matrix  $\mathbf{R}$ , we adopt the difference-based method to construct a banded estimate  $\hat{\mathbf{R}}$  of  $\mathbf{R}$ , and propose a refined estimate  $\hat{\mathbf{R}}_*^{-1}$  of  $\mathbf{R}^{-1}$ . Under mild regularity conditions, we establish consistency of  $\hat{\mathbf{R}}$  and  $\hat{\mathbf{R}}_*$  with explicit convergence rates. We also demonstrate convergence of  $\hat{\mathbf{R}}_*^{-1}$  in mean square under the  $L_\infty$  norm, though this convergence property does not hold for  $\hat{\mathbf{R}}^{-1}$ . Data-driven procedures for choosing the banding parameter and refining the estimate are developed, and simulation studies reveal their satisfactory performance. Numerical results suggest that  $\hat{\mathbf{R}}_*^{-1}$  performs well when applied to the semiparametric test statistics for detecting brain activity.

*Key words and phrases:* Autocorrelation matrix, difference-based method, fMRI, inverse, semiparametric model.

### 1. Introduction

Functional magnetic resonance imaging (fMRI) is a powerful technique to study the neural underpinnings of human cognition. It has been widely used to detect the areas of the brain responsible for certain motor or sensory tasks. An fMRI data set for a single scan on a single subject typically consists of a time series of responses for each of the voxels throughout the brain, where a voxel is a three dimensional rectangular cuboid. It is well known that fMRI data are highly temporally correlated within the same voxel and spatially correlated between different voxels (Lange (1996); Zarahn, Aguirre, and D'Esposito (1997)). A common strategy for fMRI data analysis has two major components. The first assesses the task-associated signal changes for each voxel, where adequate accommodation of the within-voxel temporal correlation is required. The second determines the regions of activation over the human brain based on the voxel-wise assessment, where the spatial correlation across voxels should be taken into consideration. We focus our attention on dealing with the temporal correlation, an essential and challenging issue for the first component.

Different statistical modeling and analysis approaches have been studied for single-voxel fMRI, including the general linear model (GLM) approach (Lange (1996); Lazar et al. (2001); Worsley et al. (2002)) and the semiparametric model approach (Zhang and Yu (2008)). For the GLM approach, ignoring the temporal correlation can result in biased test statistics for hypothesis testing of the model parameters and thus lead to inaccurate detection of the brain activity. Several methods have been proposed to handle the serial correlation with some variant of the general linear model, that can be classified into three groups (Monti (2011)): the pre-coloring method (Friston et al. (1995); Worsley and Friston (1995)), the pre-whitening method with the error term of the model assumed to be AR(1) (Bullmore et al. (1996)), AR( $m$ ) (Worsley et al. (2002)), and ARMA(1, 1) (Purdon et al. (2001)), among others, and the explicit noise modeling method (Lund et al. (2006)).

Here, we consider the semiparametric approach that yields more accurate prediction of the time-course behavior of neuronal responses than the GLM approach. For semiparametric inference, detection of the brain activity in a certain voxel can be achieved via tests of significance of the hemodynamic response function (HRF). In constructing the semiparametric test statistics, estimation of  $\mathbf{R}^{-1}$  plays an important role, where  $\mathbf{R}$  is the correlation matrix of the temporally correlated error process. A difference-based approach (Zhang et al. (2008)) was developed to estimate the autocorrelation matrix of a stationary  $g$ -dependent error process. Since the fMRI noise containing neural and nonneural sources has long-range correlations (Friston et al. (2000)), it is desirable to explore whether estimation of  $\mathbf{R}$ , under a more general assumption of the error process, can benefit from the difference-based method, which is computationally fast. Moreover, asymptotic properties of the estimates of  $\mathbf{R}$  and  $\mathbf{R}^{-1}$  need to be established for the semiparametric inference of fMRI.

We consider two classes of error processes. One is a linear process with dependence structure related to the sample size; it covers a broad range of stationary time series models, for example, the stationary ARMA( $p, q$ ) model. The other is a stationary  $g_n$ -dependent process with  $g_n$  being bounded or diverging; it includes the  $g$ -dependent process (Zhang et al. (2008); Zhang and Yu (2008)) as a special case. Under these assumptions, estimation of the Toeplitz error autocorrelation matrix  $\mathbf{R}$  is a challenging task. Directly applying the difference-based approach leads to a badly behaved estimate of  $\mathbf{R}$ , because too many autocovariances are estimated. Several regularization methods (e.g., banding, tapering, and thresholding) have been studied to yield consistent estimators for large covariance matrices in areas of high-dimensional inference (Bickel and Levina (2008a,b); Cai, Zhang, and Zhou (2010); El Karoui (2008); Furrer and Bengtsson (2007)) and time series analysis (McMurry and Politis (2010); Wu and Pourahmadi (2009);

Xiao and Wu (2012)). We adopt the regularization idea of banding and employ the difference-based approach to develop an estimate  $\widehat{\mathbf{R}}$  of  $\mathbf{R}$ . For  $\mathbf{R}^{-1}$ , we propose a refined estimate  $\widehat{\mathbf{R}}_*^{-1}$ .

Cai, Ren, and Zhou (2012), Wu and Pourahmadi (2009), and Xiao and Wu (2012) have studied the banding method for large Toeplitz covariance matrix estimation and obtained convergence rates of the estimates. Their results are not readily translated into results for our methods and the asymptotic properties of  $\widehat{\mathbf{R}}$  and  $\widehat{\mathbf{R}}_*^{-1}$  are needed. Under some mild regularity conditions, we establish explicit convergence rates of  $\widehat{\mathbf{R}}$  and  $\widehat{\mathbf{R}}_*$  and demonstrate convergence of  $\widehat{\mathbf{R}}_*^{-1}$  in mean square under the  $L_\infty$  norm. For practical guidance, data-driven approaches to choosing the banding parameter and refining the estimate of  $\mathbf{R}^{-1}$  are developed. Our contribution to the analysis of brain fMRI is in providing a computationally efficient and consistent estimate of  $\mathbf{R}^{-1}$  for semiparametric inference.

The rest of this paper is organized as follows. Section 2 reviews the semiparametric model for fMRI in Zhang and Yu (2008), and develops the banded estimate  $\widehat{\mathbf{R}}$ . Section 3 obtains explicit convergence rates of  $\widehat{\mathbf{R}}$  and  $\widehat{\mathbf{R}}_*$ , and establishes consistency of  $\widehat{\mathbf{R}}_*^{-1}$  under the  $L_\infty$  norm. Section 4 provides the data-driven methods for choosing the banding parameter and refining the estimate of  $\mathbf{R}^{-1}$ . Section 5 presents simulation evaluations of  $\widehat{\mathbf{R}}_*^{-1}$  in applying it to the semiparametric test statistics in Zhang and Yu (2008). Section 6 is illustrated with fMRI data. The Appendix includes the conditions and proofs for the main results; it is available as an online supplement.

We introduce some notation. For a matrix  $A$ ,  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  denote the smallest and largest eigenvalues of  $A$ , respectively, and  $A \succ 0$  signifies that  $A$  is positive definite. For  $A = \{a_{ij}\}$ , the  $L_1$ ,  $L_2$  and  $L_\infty$  norms are  $\|A\|_1 = \max_j \sum_i |a_{ij}|$ ,  $\|A\| \equiv \|A\|_2 = \{\lambda_{\max}(A^T A)\}^{1/2}$  and  $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ , respectively. If  $A$  is symmetric,  $\|A\|_1 = \|A\|_\infty \geq \|A\|_2$ ; see, e.g., Golub and Van Loan (1989). For a vector  $\mathbf{v} = (v_1, \dots, v_p)^T$ , the  $L_r$  norm is  $\|\mathbf{v}\|_r = (\sum_{i=1}^p |v_i|^r)^{1/r}$  for  $r \geq 1$ . Denote by  $\mathbf{z}_{p,q}$  the  $p$ th column of the  $q \times q$  identity matrix. The banding operator is defined as  $M_k(A) = \{a_{ij} \mathbf{I}(|i-j| \leq k)\}$  (Bickel and Levina (2008b)), where  $k$  is a positive integer. In the following,  $c$  and  $C$  denote generic positive constants which may vary from place to place, and do not depend on  $n$ .

## 2. Model and Method

### 2.1. Semiparametric model for single-voxel fMRI (Zhang and Yu (2008))

We briefly review the semiparametric model proposed by Zhang and Yu (2008). For single-voxel fMRI, the observations are  $\{y(t_i), s_1(t_i), \dots, s_\ell(t_i)\}_{i=1}^n$ ,

where  $y(\cdot)$  is the fMRI signal,  $s_j(t_i)$  is the external input of the  $j$ th type of stimulus at time  $t_i$ , and  $\ell$  is the number of stimulus or event types. The stimulus-related variation of the signal in a particular voxel is characterized by the hemodynamic response function (HRF). Denote by  $h_j(t)$  the HRF corresponding to the  $j$ th event type at time  $t$  after neural activity. Zhang and Yu (2008) assumed that  $h_j(t) = 0$  for  $t > t_m$ ,  $j = 1, \dots, \ell$ , where  $m$  is far below  $n$ . The semiparametric model for single-voxel fMRI is expressed as

$$\mathbf{y} = \mathbf{S}\mathbf{h} + \mathbf{d} + \boldsymbol{\epsilon}, \quad (2.1)$$

where  $\mathbf{y} = (y(t_1), \dots, y(t_n))^T$ ,  $\mathbf{S} = (\mathbf{S}_1, \dots, \mathbf{S}_\ell)$ ,  $\mathbf{h} = (\mathbf{h}_1^T, \dots, \mathbf{h}_\ell^T)^T$ ,  $\mathbf{d} = (d(t_1), \dots, d(t_n))^T$ ,  $\boldsymbol{\epsilon} = (\epsilon(t_1), \dots, \epsilon(t_n))^T$ ,  $t_i = i/n$  for  $i = 1, \dots, n$ ;  $\mathbf{S}_j$  and  $\mathbf{h}_j$  denote the  $n \times m$  Toeplitz design matrix and the  $m \times 1$  HRF vector of the  $j$ th stimulus type, respectively, for  $j = 1, \dots, \ell$ ,

$$\mathbf{S}_j = \begin{pmatrix} s_j(0) & 0 & \cdots & 0 \\ s_j(t_2 - t_1) & s_j(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_j(t_m - t_1) & s_j(t_m - t_2) & \cdots & s_j(0) \\ \vdots & \vdots & \cdots & \vdots \\ s_j(t_n - t_1) & s_j(t_n - t_2) & \cdots & s_j(t_n - t_m) \end{pmatrix}_{n \times m};$$

$\mathbf{h}_j = (h_j(t_1), \dots, h_j(t_m))^T$ ,  $d(\cdot)$  is a smooth drift function, and  $\epsilon(t)$  is a zero-mean error process with correlation matrix  $\mathbf{R}$ .

Estimation of  $\mathbf{R}^{-1}$  is essential in semiparametric inference for fMRI. Consider the hypotheses for detecting the brain activity in a certain voxel:

$$H_0 : \mathbf{U}\mathbf{h} = \mathbf{0} \quad \text{versus} \quad H_1 : \mathbf{U}\mathbf{h} \neq \mathbf{0},$$

with  $\mathbf{U}$  a full row rank matrix of size  $K_0 \times \ell m$  and  $K_0$  a positive integer. Zhang and Yu (2008) proposed the semiparametric test statistic  $\mathbb{K} = (\mathbf{U}\hat{\mathbf{h}})^T \{ \mathbf{U}(\tilde{\mathbf{S}}^T \hat{\mathbf{R}}^{-1} \tilde{\mathbf{S}})^{-1} \mathbf{U}^T \}^{-1} (\mathbf{U}\hat{\mathbf{h}}) / \{ \hat{\mathbf{r}}^T \hat{\mathbf{R}}^{-1} \hat{\mathbf{r}} / (n - \ell m) \}$  and the bias-corrected version  $\mathbb{K}_{bc} = (\mathbf{U}\hat{\mathbf{h}}_{bc})^T \{ \mathbf{U}(\tilde{\mathbf{S}}^T \hat{\mathbf{R}}^{-1} \tilde{\mathbf{S}})^{-1} \mathbf{U}^T \}^{-1} (\mathbf{U}\hat{\mathbf{h}}_{bc}) / \{ \hat{\mathbf{r}}_{bc}^T \hat{\mathbf{R}}^{-1} \hat{\mathbf{r}}_{bc} / (n - \ell m) \}$ , where  $\hat{\mathbf{R}}^{-1}$  is the estimate of  $\mathbf{R}^{-1}$ . They further demonstrated that under  $H_0$ ,  $\mathbb{K} \xrightarrow{D} \chi_{K_0}^2$  and  $\mathbb{K}_{bc} \xrightarrow{D} \chi_{K_0}^2$ , assuming some regularity conditions of which one is

$$E(\|\hat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty^2) = o(1); \quad (2.2)$$

see Condition A8 therein. We develop a consistent estimate of  $\mathbf{R}$  and a refined estimate of  $\mathbf{R}^{-1}$  that fulfills (2.2).

**2.2. Assumption of the error process**

It's known that the error process consisting of physical and physiological processes (Lund et al. (2006)) is highly temporally correlated. Assumptions on the error process have included AR(1) (Bullmore et al. (1996)), AR( $m$ ) (Worsley et al. (2002)), ARMA(1, 1) (Purdon et al. (2001)), and  $g$ -dependence (Zhang et al. (2008); Zhang and Yu (2008)). We consider linear processes and  $g_n$ -dependent processes for the error.

Suppose  $\{\epsilon(t_i)\}$  is a zero-mean linear process (see, e.g., Shumway and Stoffer (2011)) with dependence structure related to  $n$ . As in Condition A1 in the Appendix,

$$\epsilon(t_i) = \sum_{j=-\infty}^{\infty} \phi_{n;j} w_{i-j},$$

where (i)  $\phi_{n;0} \equiv 1$ ; (ii) for any  $n \geq 1$ , there exists  $1 \leq g_n \leq n - 1$  and  $\alpha_n > 4$ , such that

$$|\phi_{n;j}| \leq \begin{cases} C, & \text{if } |j| \leq \frac{g_n}{2}, \\ C|2j|^{-\alpha_n}, & \text{if } |j| > \frac{g_n}{2}, \end{cases}$$

with a constant  $C > 0$ ; (iii)  $\{w_i\}_{i=-\infty}^{\infty}$  is a sequence of independent white noises (WN) with  $E(w_i) = 0$ ,  $E(w_i^2) = \sigma_w^2$ , and  $\sup_i E(w_i^4) < \infty$ .

Assumption (ii) illustrates the decay rate of the coefficients, and  $\alpha_n > 4$  guarantees that  $\sum_{j=-\infty}^{\infty} |\phi_{n;j}| \leq Cg_n$ . From (iii), the error process has zero mean and a finite fourth moment for any fixed  $n$ , which is weaker than the Gaussian tail (Cai, Ren, and Zhou (2012)) and higher order of polynomial tail (Xiao and Wu (2012)) assumptions. Condition A1 is satisfied by, for example, the zero-mean stationary ARMA( $p, q$ ) model with a finite fourth moment and independent white noises, for any  $1 \leq g_n \leq n - 1$  and any fixed  $\alpha_n > 4$ .

Motivated by the  $g$ -dependence assumption in Zhang et al. (2008) and Zhang and Yu (2008), we consider another class of error process with  $\epsilon$  a zero-mean stationary  $g_n$ -dependent sequence, where  $g_n$  can be bounded or diverge with  $n$ , see Condition B1 in the Appendix. This assumption includes the  $g$ -dependent process (Zhang et al. (2008); Zhang and Yu (2008)) as a special case. There is overlap between A1 and B1, for example the zero-mean MA( $g_n$ ) process with a finite fourth moment and independent white noises satisfies both.

Under A1 or B1, the true autocorrelation matrix of  $\{\epsilon(t_i)\}$  is Toeplitz,

$$\mathbf{R} = \begin{pmatrix} 1 & \rho(1) & \cdots & \rho(n-1) \\ \rho(1) & 1 & \cdots & \rho(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(n-1) & \rho(n-2) & \cdots & 1 \end{pmatrix},$$

where  $\rho(k) = \gamma(k)/\gamma(0)$ , for  $k = 1, \dots, n-1$  and  $\gamma(k) = \text{cov}\{\epsilon(t_i), \epsilon(t_{i+k})\}$  is the autocovariance.

**Remark 1.** Under Condition A1,

$$\gamma(k) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \phi_{n;j} \phi_{n;j+|k|}, \quad k = 0, \pm 1, \dots, \pm(n-1). \quad (2.3)$$

From (2.3), we can show that  $\sigma_w^2 \leq \gamma(0) \leq Cg_n$ , and

$$|\gamma(k)| \leq \begin{cases} C(g_n + 1 - k), & \text{if } 0 \leq k \leq g_n, \\ C(2k - g_n)^{1-\alpha_n}, & \text{if } g_n < k \leq n-1. \end{cases}$$

Therefore, A1 is significantly different from the assumptions in Cai, Ren, and Zhou (2012) and Xiao and Wu (2012), where the dependence structures of the time series are not related to  $n$ . Under B1,  $\gamma(k) = 0$  for  $k > g_n$ .

Under A1 then,

$$|\rho(k)| \leq \begin{cases} \frac{C(g_n + 1 - k)}{\gamma(0)}, & \text{if } 0 \leq k \leq g_n, \\ \frac{C(2k - g_n)^{1-\alpha_n}}{\gamma(0)}, & \text{if } g_n < k \leq n-1, \end{cases}$$

so  $\mathbf{R}$  is a ‘‘bandable’’ matrix where the entries decay as they move away from the diagonal. The decay rate of  $|\rho(k)|$  for  $k > g_n$  is much faster than that for  $k \leq g_n$ . Thus,  $g_n$  could be regarded as the natural banding parameter of  $\mathbf{R}$ . Under B1,  $\mathbf{R}$  is a banded matrix with  $g_n$  the exact banding parameter.

### 2.3. Estimation of the error autocorrelation matrix

Several methods have been proposed to estimate large Toeplitz covariance matrices, see Cai, Ren, and Zhou (2012), McMurry and Politis (2010), Wu and Pourahmadi (2009), and Xiao and Wu (2012), among others. Essentially, they consider regularizing the sample covariance matrix, which is directly constructed from the observed time series. However, for fMRI data, the error process is unobservable. To obtain the error autocovariance estimates, we adopt the difference-based method (Zhang et al. (2008)), that is computationally fast. Since  $\mathbf{R}$  is ‘‘bandable’’ under either of the two conditions for the error process, we construct a  $g_n$ -banded estimate.

At (2.1), we take the second-order difference of  $(\mathbf{y} - \mathbf{S}\mathbf{h})$  and get

$$\mathbf{e} \equiv \mathbf{D}_2(\mathbf{y} - \mathbf{S}\mathbf{h}) = \mathbf{D}_2(\boldsymbol{\epsilon} + \mathbf{d}), \quad (2.4)$$

where  $\mathbf{D}_2$  is the  $(n - 2) \times n$  matrix

$$\mathbf{D}_2 = \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-2) \times n}.$$

Here,  $e(t_i) = \{\epsilon(t_i) - 2\epsilon(t_{i-1}) + \epsilon(t_{i-2})\} + \{d(t_i) - 2d(t_{i-1}) + d(t_{i-2})\}$ , for  $i = 3, \dots, n$ , where  $e(t_i) = \mathbf{z}_{i-2, n-2}^T \mathbf{e}$  and  $\mathbf{z}_{p,q}$  is the  $p$ th column of the  $q \times q$  identity matrix. Let  $\gamma_e(k) = \text{cov}\{e(t_i), e(t_{i+k})\}$  for  $0 \leq k \leq n - 3$ . Then

$$\gamma_e(k) = \gamma(k - 2) - 4\gamma(k - 1) + 6\gamma(k) - 4\gamma(k + 1) + \gamma(k + 2), \quad k = 0, \dots, n - 3,$$

which implies

$$\begin{pmatrix} \gamma_e(0) \\ \vdots \\ \gamma_e(k) \end{pmatrix} = A_k \begin{pmatrix} \gamma(0) \\ \vdots \\ \gamma(k) \end{pmatrix} + B_k \begin{pmatrix} \gamma(k + 1) \\ \gamma(k + 2) \end{pmatrix}, \quad k = 0, \dots, n - 3, \quad (2.5)$$

where  $A_k$  is a  $(k + 1) \times (k + 1)$  matrix

$$A_k = \begin{pmatrix} 6 & -8 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 7 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}_{(k+1) \times (k+1)},$$

and  $B_k$  is a  $(k + 1) \times 2$  matrix, satisfying

$$B_0 = (-8, 2), \quad B_1 = \begin{pmatrix} 2 & 0 \\ -4 & 1 \end{pmatrix}, \quad B_k^T = \begin{pmatrix} 0 & \cdots & 0 & 1 & -4 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad k \geq 2.$$

The basic idea of the difference-based method is to first obtain the empirical estimates of  $\gamma_e(k)$  for  $k = 0, \dots, g_n$ , and then, by (2.5), acquire the estimates of  $\gamma(k)$  for  $k = 0, \dots, g_n$ , which enables us to construct the error autocorrelation matrix estimate. A two-step procedure describes in detail how to build a  $g_n$ -banded estimate of  $\mathbf{R}$ .

**Step I.** An initial estimate of  $\mathbf{h}$  is constructed based on the first-order difference of  $\mathbf{y}$ .

Let  $\mathbf{D}_1$  be an  $(n - 1) \times n$  matrix

$$\mathbf{D}_1 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{(n-1) \times n}.$$

The first-order difference of  $\mathbf{y}$  yields

$$\mathbf{D}_1\mathbf{y} = \mathbf{D}_1\mathbf{S}\mathbf{h} + \mathbf{D}_1\mathbf{d} + \mathbf{D}_1\boldsymbol{\epsilon}. \quad (2.6)$$

By smoothness of the drift function in A3,  $d(t_i) - d(t_{i-1}) = d'(t_{i-1})n^{-1} + O(n^{-2}) = O(n^{-1})$ , and the term  $\mathbf{D}_1\mathbf{d}$  in (2.6) can be ignored. Ordinary least-squares method provides an initial estimate of the HRF,

$$\hat{\mathbf{h}}_{\text{DBE}} = \{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{y}).$$

**Step II.** Construct the  $g_n$ -banded estimate of  $\mathbf{R}$  based on the empirical estimates of  $\gamma_e(k)$ , for  $k = 0, \dots, g_n$ .

Substituting  $\hat{\mathbf{h}}_{\text{DBE}}$  for  $\mathbf{h}$  in (2.4), we get  $\hat{\mathbf{e}} \equiv \mathbf{D}_2(\mathbf{y} - \mathbf{S}\hat{\mathbf{h}}_{\text{DBE}})$ . The empirical estimate of  $\gamma_e(k)$  is

$$\hat{\gamma}_e(k) = \frac{1}{n} \sum_{i=3}^{n-k} \hat{e}(t_i)\hat{e}(t_{i+k}), \quad k = 0, \dots, g_n, \quad (2.7)$$

where  $\hat{e}(t_i) = \mathbf{z}_{i-2, n-2}^T \hat{\mathbf{e}}$ , for  $i = 3, \dots, n$ . It is also used by Wu and Pourahmadi (2009). By taking  $k = g_n$  in (2.5),

$$\boldsymbol{\gamma}_e = A_{g_n}\boldsymbol{\gamma} + B_{g_n}(\gamma(g_n+1) \ \gamma(g_n+2))^T, \quad (2.8)$$

where  $\boldsymbol{\gamma}_e = (\gamma_e(0), \dots, \gamma_e(g_n))^T$  and  $\boldsymbol{\gamma} = (\gamma(0), \dots, \gamma(g_n))^T$ . Since  $\gamma(g_n+1)$  and  $\gamma(g_n+2)$  are negligible, as shown in Remark 1, we drop the second term on the right hand side of (2.8) and estimate  $\boldsymbol{\gamma}$  by

$$\hat{\boldsymbol{\gamma}} = A_{g_n}^{-1}\hat{\boldsymbol{\gamma}}_e, \quad (2.9)$$

where  $\hat{\boldsymbol{\gamma}}_e = (\hat{\gamma}_e(0), \dots, \hat{\gamma}_e(g_n))^T$ . The estimate of  $\rho(k)$  is derived as

$$\hat{\rho}(k) = \begin{cases} \hat{\gamma}(k), & \text{if } 0 \leq k \leq g_n, \\ \hat{\gamma}(0), & \text{if } g_n < k \leq n-1, \\ 0, & \text{if } g_n < k \leq n-1, \end{cases}$$

where  $\hat{\gamma}(k) = \mathbf{z}_{k+1, g_n+1}^T \hat{\boldsymbol{\gamma}}$ , and the  $g_n$ -banded estimate of  $\mathbf{R}$  is

$$\hat{\mathbf{R}} = \begin{pmatrix} 1 & \hat{\rho}(1) & \cdots & \hat{\rho}(n-1) \\ \hat{\rho}(1) & 1 & \cdots & \hat{\rho}(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}(n-1) & \hat{\rho}(n-2) & \cdots & 1 \end{pmatrix}. \quad (2.10)$$

This estimation procedure is for single-run fMRI. If we have a fixed number of replications, the estimate of the autocorrelation matrix can be acquired as follows.



Let  $(\mathbf{y}^1, \mathbf{S}^1), \dots, (\mathbf{y}^N, \mathbf{S}^N)$  be the observations, where  $\mathbf{y}^i$  and  $\mathbf{S}^i$  are the vector of response and the design matrix for the  $i$ th realization, and  $N$  is a fixed integer denoting the number of replications. From Step I above, we obtain the initial estimates of the HRFs,  $\hat{\mathbf{h}}_{\text{DBE}}^i = \{(\mathbf{D}_1 \mathbf{S}^i)^T (\mathbf{D}_1 \mathbf{S}^i)\}^{-1} (\mathbf{D}_1 \mathbf{S}^i)^T (\mathbf{D}_1 \mathbf{y}^i)$  for  $i = 1, \dots, N$ . The empirical estimates  $\hat{\gamma}_e^i$  is based on  $\mathbf{D}_2(\mathbf{y}^i - \mathbf{S}^i \hat{\mathbf{h}}_{\text{DBE}}^i)$  following Step II. We estimate  $\gamma_e$  by

$$\hat{\gamma}_e = \frac{1}{N} \sum_{i=1}^N \hat{\gamma}_e^i. \tag{2.11}$$

Thereafter, the same approaches as in Step II lead to  $\hat{\mathbf{R}}$ .

The estimate (2.11) parallels the one proposed in Cai, Ren, and Zhou (2012). In the following, we focus on single-run fMRI.

### 3. Theoretical Results

We establish an explicit convergence rate of  $\hat{\mathbf{R}}$ , develop a positive-definite estimate of  $\mathbf{R}^{-1}$ , and demonstrate consistency of the estimate under the  $L_\infty$  norm. The conditions and proofs are given in the Appendix, available on the online supplement.

**Theorem 1.** *Assume A2–A5 in the Appendix. Under either of the following two assumptions:*

1. A1 in the Appendix holds,  $n = o(g_n^{8+2\alpha_n} + 2^{\alpha_n})$  and  $g_n^{14}/n = o(1)$ ,
2. B1 in the Appendix holds and  $g_n^{14}/n = o(1)$ ,

for  $\hat{\mathbf{R}}$  in (2.10),  $\|\hat{\mathbf{R}} - \mathbf{R}\|_\infty = O_P(g_n^7/n^{1/2})$ .

In Theorem 1, the banded estimate  $\hat{\mathbf{R}}$  is consistent under the  $L_\infty$  norm with convergence rate  $g_n^7/n^{1/2}$ . Recall that  $\gamma$  is estimated by  $\hat{\gamma} = A_{g_n}^{-1} \hat{\gamma}_e$ , as in (2.9). In Lemma 1 in the Appendix, we show that  $A_{g_n}^{-1}$  exists for any  $g_n \geq 1$  and  $\|A_{g_n}^{-1}\|_1 = O(g_n^4)$  as  $n \rightarrow \infty$ , the major reason for the power of  $g_n$  in the convergence rate. It is the price we have to pay for the difference-based method in order to gain computational efficiency. From Theorem 1, we obtain the same convergence rate for  $\hat{\mathbf{R}}$  under either the linear process assumption or the  $g_n$ -dependence assumption for the error process.

For the ARMA( $p, q$ ) model that satisfies A1, we can take  $g_n$  to satisfy the rate assumptions in Theorem 1 with any constant-valued  $\alpha_n > 4$ .

**Remark 2.** In Xiao and Wu (2012), the optimal convergence rate of the banded autocovariance matrix estimate of stationary process is obtained. We compare our convergence rate with theirs under the following conditions. Suppose that  $\{\epsilon(t_i)\}$  are observed and  $\epsilon(t_i) = \sum_{j=0}^\infty \phi_j w_{i-j}$ , where (i)  $\phi_0 = 1$ , (ii)  $|\phi_j| \leq$

$C|2j|^{-\alpha_n}$ , for  $j \neq 0$ , with  $\alpha_n \equiv \alpha_0 + 1$ ,  $\alpha_0 > 3$ , (iii)  $\{w_i\}_{i=-\infty}^{\infty}$  are i.i.d. white noises with  $E(w_i) = 0$ ,  $E(w_i^2) = \sigma_w^2$  and  $E(w_i^p) < \infty$  for a constant  $p > 4$ .

In Xiao and Wu (2012), the convergence rate of the banded estimate under the  $L_2$  norm is  $\{\log(n)/n\}^{\alpha_0/(2\alpha_0+1)}$ . For our convergence rate, the assumption  $n = o(g_n^{8+2\alpha_n} + 2^{\alpha_n})$  implies  $(1/n)^{(\alpha_0-2)/(2\alpha_0+10)} = o(g_n^7/n^{1/2})$ . Since  $\alpha_0/(2\alpha_0 + 1) > (\alpha_0 - 2)/(2\alpha_0 + 10)$ , the convergence rate in Xiao and Wu (2012) is sharper. One important reason is that they considered the matrix  $L_2$  norm in order to make use of the spectral density of the stationary process, while we use the  $L_\infty$  norm which is not smaller than the  $L_2$  norm.

**Proposition 1.** *Assume A1–A5 in the Appendix. If  $\alpha_n g_n \rightarrow \infty$  and  $g_n^{14}/n = o(1)$ , for  $\widehat{\mathbf{R}}$  in (2.10),  $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty = o_P(1)$ .*

**Proposition 2.** *Assume A2–A6 in the Appendix. Under either of the following two assumptions:*

1. A1 in the Appendix holds,  $\alpha_n g_n \rightarrow \infty$  and  $g_n^{14}/n = o(1)$ ,
2. B1 in the Appendix holds and  $g_n^{14}/n = o(1)$ ,

for  $\widehat{\mathbf{R}}$  in (2.10),  $\lim_{n \rightarrow \infty} P(\widehat{\mathbf{R}} \succ 0) = 1$ .

While Proposition 2 has  $\widehat{\mathbf{R}}$  positive definite with probability tending to one, in finite-sample situations,  $\widehat{\mathbf{R}}$  may not be. To obtain positive-definite estimates, several modification methods have been proposed (e.g., Cai and Zhou (2012); Wu and Xiao (2011); Zhang et al. (2008); Zhang and Yu (2008)). Zhang et al. (2008) and Zhang and Yu (2008) proposed a positive-definite estimate of  $\mathbf{R}^{-1}$  by refining  $\widehat{\mathbf{R}}^{-1}$ . Their estimate is

$$\widehat{\mathbf{R}}_Z^{-1} = \begin{cases} \widehat{\mathbf{R}}^{-1}, & \text{if } \widehat{\mathbf{R}} \succ 0, \\ \mathbf{I}_n, & \text{otherwise,} \end{cases}$$

where  $\mathbf{I}_n$  is the  $n \times n$  identity matrix and  $\widehat{\mathbf{R}}$  is constructed with  $g_n = 2$ . Motivated by Cai and Zhou (2012), we propose an estimate of  $\mathbf{R}^{-1}$  by refining  $\widehat{\mathbf{R}}_Z^{-1}$ ,

$$\widehat{\mathbf{R}}_*^{-1} = \begin{cases} \widehat{\mathbf{R}}^{-1}, & \text{if } \widehat{\mathbf{R}} \succ 0 \text{ and } \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega, \\ \mathbf{I}_n, & \text{otherwise,} \end{cases} \quad (3.1)$$

where  $0 < \omega < 1/2$  and  $D > 0$  are constants.

When  $\widehat{\mathbf{R}}$  is not positive definite, we estimate  $\mathbf{R}^{-1}$  by  $\mathbf{I}_n$ . This is reasonable, because it is the true autocorrelation matrix when the error process is uncorrelated. Even if  $\widehat{\mathbf{R}}$  is positive definite, it is possible that the eigenvalues of  $\widehat{\mathbf{R}}$  are close to 0, resulting in a large scale of  $\|\widehat{\mathbf{R}}^{-1}\|_\infty$ . In order to exclude the case that  $\widehat{\mathbf{R}}$  is almost singular, we set a threshold on  $\|\widehat{\mathbf{R}}^{-1}\|_\infty$ . Thus,  $\widehat{\mathbf{R}}_*^{-1}$  is a positive-definite estimate of  $\mathbf{R}^{-1}$  with the  $L_\infty$  norm bounded by  $\max\{Dn^\omega, 1\}$ . As an estimate of  $\mathbf{R}$ ,  $\widehat{\mathbf{R}}_*$  could be viewed as a refined version of  $\widehat{\mathbf{R}}$  and  $\widehat{\mathbf{R}}_Z$ .

**Proposition 3.** *Under A6 in the Appendix and the conditions of Theorem 1,*

$$\|\widehat{\mathbf{R}}_Z - \mathbf{R}\|_\infty = O_P(g_n^7/n^{1/2}), \quad \|\widehat{\mathbf{R}}_* - \mathbf{R}\|_\infty = O_P(g_n^7/n^{1/2}).$$

**Theorem 2.** *Assume A2–A6 in the Appendix. Under either of the following two assumptions:*

1. A1 in the Appendix holds,  $\alpha_n g_n \rightarrow \infty$  and  $g_n^{14}/n^{1-2\omega} = o(1)$ ,
2. B1 in the Appendix holds and  $g_n^{14}/n^{1-2\omega} = o(1)$ ,

for  $\widehat{\mathbf{R}}_*^{-1}$  defined in (3.1), we have

$$\lim_{n \rightarrow \infty} E(\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2) = 0.$$

This result has  $\widehat{\mathbf{R}}_*^{-1}$  fulfilling Condition A8 in Zhang and Yu (2008) under either of the two conditions for the error process. Hence,  $\widehat{\mathbf{R}}_*^{-1}$  offers a satisfactory estimate of  $\mathbf{R}^{-1}$  for the semiparametric inference for brain fMRI data. For  $\widehat{\mathbf{R}}_Z^{-1}$ , the convergence property described in Theorem 2 also holds under certain conditions, e.g.,  $0 < c \leq \lambda_{\min}(\widehat{\mathbf{R}}_Z) \leq \lambda_{\max}(\widehat{\mathbf{R}}_Z) < C$  or  $\|\widehat{\mathbf{R}}_Z^{-1}\|_\infty \leq C$ . These conditions are relaxed here as  $\|\widehat{\mathbf{R}}_*^{-1}\|_\infty$  is allowed to diverge with  $n$ .

**Proposition 4.** *Under A6 in the Appendix and the conditions of Theorem 1,*

$$\begin{aligned} \|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty &= O_P\left(\frac{g_n^7}{n^{1/2}}\right), \quad \|\widehat{\mathbf{R}}_Z^{-1} - \mathbf{R}^{-1}\|_\infty = O_P\left(\frac{g_n^7}{n^{1/2}}\right), \\ \|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty &= O_P\left(\frac{g_n^7}{n^{1/2}}\right). \end{aligned}$$

#### 4. Data-driven Methods for $g_n$ and $\widehat{\mathbf{R}}_*^{-1}$

##### 4.1. Choosing the banding parameter $g_n$

Results in Section 3 give rates of  $g_n$  that lead to consistency of  $\widehat{\mathbf{R}}$  and  $\widehat{\mathbf{R}}_*^{-1}$ , but they do not give much guidance for the choice of  $g_n$  in practice. To select the banding parameter, we utilize the idea of risk-minimization (Bickel and Levina (2008b)) and the technique of subsampling (Politis, Romano, and Wolf (1999)).

Since the estimates of  $\gamma_e(k)$  and  $\gamma(k)$  do not perform well when  $k$  is close to  $n$ , we select  $g_n$  from  $\{0, \dots, T\}$  instead of  $\{0, \dots, n - 1\}$ , where  $T$  is a pre-determined integer much less than  $(n - 1)$ . In the theoretical analysis, we took  $1 \leq g_n \leq n - 1$ , but in applications, one needs to consider the case of  $g_n = 0$ . Let  $\boldsymbol{\gamma}_{e,u} = (\gamma_e(0), \dots, \gamma_e(u), 0, \dots, 0)^T$  and  $\boldsymbol{\gamma}_u = (\gamma(0), \dots, \gamma(u), 0, \dots, 0)^T$  be vectors of length  $T+1$ , for  $u = 0, \dots, T$ . Denote by  $\widehat{\boldsymbol{\gamma}}_{e,u} = (\widehat{\gamma}_e(0), \dots, \widehat{\gamma}_e(u), 0, \dots, 0)^T$  the estimate of  $\boldsymbol{\gamma}_{e,u}$ , with  $\widehat{\gamma}_e(k)$  calculated in (2.7) and by  $\widehat{\boldsymbol{\gamma}}_u = (\widehat{\gamma}(0), \dots, \widehat{\gamma}(u), 0, \dots, 0)^T$  the estimate of  $\boldsymbol{\gamma}_u$ , satisfying  $(\widehat{\gamma}(0), \dots, \widehat{\gamma}(u))^T = A_u^{-1}(\widehat{\gamma}_e(0), \dots, \widehat{\gamma}_e(u))^T$ . Based on the risk-minimization method, we develop a two-step procedure to select  $g_n$ .

**Step 1.** The initial choice of the banding parameter  $g_e$  is that which minimizes the risk

$$r_e(g) = E(\|\widehat{\gamma}_{e,g} - \gamma_{e,T}\|_1), \quad (4.1)$$

for  $g \in \{2, \dots, T\}$ .

We take  $g_e$  not less than 2 because  $\mathbf{e}$ , defined in (2.4), is the second-order difference of  $\boldsymbol{\epsilon} + \mathbf{d}$ . For the simplest case, if  $\boldsymbol{\epsilon}$  is a sequence of white noises,  $\mathbf{e}$  would be 2-dependent.

**Step 2.** Select the banding parameter as the minimizer of

$$r(g) = E(\|\widehat{\gamma}_g - \gamma_{g_e}\|_1), \quad (4.2)$$

for  $g \in \{0, \dots, g_e\}$ .

Since the Toeplitz autocorrelation matrix is uniquely determined by the autocovariances, our risk functions (4.1) and (4.2) parallel the one in Wu and Pourahmadi (2009), which is constructed based on the Toeplitz autocovariance matrix and its estimate.

We subsample to estimate the risk functions. For  $\{\widehat{e}(t_i)\}_{i=3}^n$ , blocks of consecutive elements are used as legitimate subsamples. The length of each block is  $b$ , predetermined in practice. Therefore, there are  $(n - b - 1)$  blocks, and the  $\nu$ th one is  $\{\widehat{e}(t_{\nu+2}), \widehat{e}(t_{\nu+3}), \dots, \widehat{e}(t_{\nu+b+1})\}$ , for  $\nu = 1, \dots, n - b - 1$ . We select  $V$  subsamples, denoted by  $G_1, \dots, G_V$ , such that  $G_\mu = \{\widehat{e}(t_{(\mu-1)\lfloor (n-b-2)/(V-1)\rfloor+3}), \dots, \widehat{e}(t_{(\mu-1)\lfloor (n-b-2)/(V-1)\rfloor+b+2})\}$ , for  $\mu = 1, \dots, V$ , where  $\lfloor \cdot \rfloor$  denotes the floor function. To enhance the computational efficiency, only  $G_1, \dots, G_V$  are used. Denote by  $\widehat{\gamma}_{e,u}^\nu = (\widehat{\gamma}_e^\nu(0), \dots, \widehat{\gamma}_e^\nu(u), 0, \dots, 0)^T$  and  $\widehat{\gamma}_u^\nu = (\widehat{\gamma}^\nu(0), \dots, \widehat{\gamma}^\nu(u), 0, \dots, 0)^T$  the estimates of  $\gamma_{e,u}$  and  $\gamma_u$  based on the subsample  $G_\nu$ , respectively.

For (4.1), alternatively, each of  $G_1, \dots, G_V$  is used to estimate  $\gamma_{e,T}$ , which serves as the “target”, and the other  $(V - 1)$  blocks are used to estimate  $\gamma_{e,g}$  for  $g = 2, \dots, T$ . The “target” is a vector of length  $T + 1$ , with  $T$  much smaller than  $n$ , so the number of estimated parameters in the “target” is much reduced and we use just one subsample to estimate the “target”. The estimated risk is

$$\widehat{r}_e(g) = \frac{1}{V(V-1)} \sum_{\nu=1}^V \sum_{\mu=1, \mu \neq \nu}^V \|\widehat{\gamma}_{e,g}^\mu - \widehat{\gamma}_{e,T}^\nu\|_1.$$

The initial choice  $\widehat{g}_e$  is the minimizer of  $\widehat{r}_e(g)$  for  $g \in \{2, \dots, T\}$ . Similarly, we estimate  $r(g)$  by

$$\widehat{r}(g) = \frac{1}{V(V-1)} \sum_{\nu=1}^V \sum_{\mu=1, \mu \neq \nu}^V \|\widehat{\gamma}_g^\mu - \widehat{\gamma}_{\widehat{g}_e}^\nu\|_1,$$

and the banding parameter is selected as  $\widehat{g} = \arg \min_{0 \leq g \leq \widehat{g}_e} \widehat{r}(g)$ .

### 4.2. Choosing $D$ and $\omega$ for $\widehat{\mathbf{R}}_*^{-1}$

To calculate  $\widehat{\mathbf{R}}_*^{-1}$ , we develop data-driven methods for the choices of  $D > 0$  and  $0 < \omega < 1/2$ . After  $\widehat{g}$  is selected, if  $\widehat{\mathbf{R}}$  is not positive definite, from (3.1),  $\mathbf{I}_n$  is used to estimate  $\mathbf{R}^{-1}$  and there is no need to choose  $D$  and  $\omega$ . Therefore, for the choices of  $D$  and  $\omega$ , we only consider that  $\widehat{\mathbf{R}} \succ 0$  which implies that  $\widehat{\mathbf{R}}^{-1}$  exists. In finite-sample situations, we can fix  $\omega$  to be  $1/2$ , since simulation results (not shown) reveal that for different values of  $0 < \omega \leq 1/2$ , with a proper choice of  $D$ , the calculated  $\widehat{\mathbf{R}}_*^{-1}$  are not significantly different. In the following, we will choose  $D$  from a proper interval on  $\mathbb{R}^+$ .

Following the procedure in Section 4.1, we take subsamples  $G_1, \dots, G_V$ . Based on  $G_\nu$ ,  $\nu = 1, \dots, V$ , denote by  $\widehat{\mathbf{R}}_\nu$  the estimate of  $\mathbf{R}$ , and then the refined estimate of  $\mathbf{R}^{-1}$  is

$$\widehat{\mathbf{R}}_{*;\nu,D}^{-1} = \begin{cases} \widehat{\mathbf{R}}_\nu^{-1}, & \text{if } \widehat{\mathbf{R}}_\nu \succ 0, \text{ and } \|\widehat{\mathbf{R}}_\nu^{-1}\|_\infty \leq Dn^{1/2}, \\ \mathbf{I}_n, & \text{otherwise.} \end{cases}$$

Then,  $\widehat{D}$  is selected as the minimizer of

$$\widehat{r}_*(D) = \frac{1}{V} \sum_{\nu=1}^V \|\widehat{\mathbf{R}}_{*;\nu,D}^{-1} - \widehat{\mathbf{R}}^{-1}\|_\infty,$$

by grid search method.

## 5. Numerical Results

We report on simulation studies to evaluate the data-driven methods in Section 4, check the performance of  $\widehat{\mathbf{R}}_*^{-1}$  by applying it to the test statistics  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  (Zhang and Yu (2008)) and compare the performance of  $\widehat{\mathbf{R}}_*$  and  $\widehat{\mathbf{R}}_Z$  in Zhang and Yu (2008).

The hypotheses  $H_0 : \mathbf{h} = \mathbf{0}$  versus  $H_1 : \mathbf{h} \neq \mathbf{0}$  are considered to test whether there's neural activity in a voxel or not. We generated data for a single run with  $n = 400$ . According to the null hypothesis,  $h_j(t_i) = 0$  for  $i = 1, \dots, m$ ,  $j = 1, \dots, \ell$ . We used  $d(t_i) = 10 \sin\{\pi(t_i - 0.21)\}$  as the drift function. Three types of errors were studied: MA(4), ARMA(1, 3), and the sum of AR(1) and a white noise. For each type of error, we considered both a single event type ( $\ell = 1$ ) with  $m = 20$ , and two event types ( $\ell = 2$ ) with  $m = 15$ . Stimuli were generated as follows: for  $\ell = 1$ ,  $\{s_1(t_i)\}$  are independent Bernoulli(0.5); for  $\ell = 2$ ,  $\{s_1(t_i), s_2(t_i), 1 - s_1(t_i) - s_2(t_i)\}$  are independent from the trinomial distribution (1; 1/3, 1/3, 1/3). For the local linear smoothing matrix  $S_d$  used in  $\mathbb{K}$  and  $\mathbb{K}_{bc}$ , we used the Epanechnikov kernel function; see (3.3) in Zhang and Yu (2008) for details. The bandwidth parameter of  $S_d$  was selected by the ‘‘pWPL’’ method proposed in Zhang et al. (2008).

The methods of choosing the banding parameter  $g_n$  and selecting  $D$  and  $\omega$  for  $\widehat{\mathbf{R}}_*^{-1}$  are illustrated in Section 4. In our simulation study, we took  $V = 20$ . For the block size  $b$ , as a general guide, Politis, Romano, and Wolf (1999) showed that  $b$  should grow to infinity while  $b/n \rightarrow 0$  with a rate  $n^{1/3}$ ; we took  $b = \lfloor C_b n^{1/3} \rfloor$  where  $C_b > 0$  is a constant. Simulation results (not shown) reveal that smaller values of  $C_b$  correspond to underestimating the banding parameter and larger values of  $C_b$  result in larger estimates. The choice of  $C_b = 8$  has overall good performance in selecting the banding parameter for different models of the error process; we took  $b = \lfloor 8n^{1/3} \rfloor$  in the calculation. The choice of  $T$  is also important in practice. One can use  $T \leq n/4$  as in Box and Jenkins (1976), but the default value in R is  $T = 10 \log(10n)$ ; we use  $T = \lfloor C_T \log(10n) \rfloor$  with a positive constant  $C_T$ , provided  $T < b$ . Simulation results (not shown) show that the choice of  $C_T$  has no significant impact on the selection of the banding parameter. Taking computational efficiency into consideration, we fix  $C_T = 3$  in practice, so  $T = \lfloor 3 \log(10n) \rfloor$ .

To investigate the performance of  $\widehat{\mathbf{R}}_*^{-1}$ , we carried out 500 realizations in each simulation study and drew a QQ plot of the empirical (1st to 99th) percentiles of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  versus the theoretical percentiles of the  $\chi_{\ell m}^2$  distribution. We also compared the performance of  $\widehat{\mathbf{R}}_*$  and  $\widehat{\mathbf{R}}_Z$  in Zhang and Yu (2008).

### 5.1. Example 1: MA(4) error process

The error process was generated by  $\epsilon(t_i) = z(t_i) + \theta_1 z(t_{i-1}) + \theta_2 z(t_{i-2}) + \theta_3 z(t_{i-3}) + \theta_4 z(t_{i-4})$ , where  $\theta_1 = 0.75$ ,  $\theta_2 = 0.5$ ,  $\theta_3 = 0.25$  and  $\theta_4 = 0.35$ . For a single event type,  $\{z(t_i)\}$  were independent normal with mean 0 and variances  $0.4786^2$  and  $0.4786^2/8$ , respectively. These choices give the signal-to-noise ratios (SNRs) of about 1 and 8, respectively, where  $\text{SNR} = \text{var}(\mathbf{Sh})/\text{var}(\boldsymbol{\epsilon})$ . For two event types,  $\{z(t_i)\}$  had the same distribution but with variances  $0.4575^2$  and  $0.4575^2/8$ , corresponding to SNRs about 1 and 8.

In order to see the importance of estimating the autocorrelation matrix  $\mathbf{R}$ , Figure 1 shows QQ plots of the percentiles of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  against those of the  $\chi_{\ell m}^2$  distribution. For the test statistics calculated in Figure 1,  $\mathbf{R}$  was estimated by  $\mathbf{I}_n$ , which means the autocorrelation structure of the error process was ignored. The test statistics calculated in the top panels are for a single event type and those in the bottom panels are for two event types. From Figure 1, the finite sampling distributions of the test statistics calculated with  $\mathbf{R}$  estimated by  $\mathbf{I}_n$  do not show reasonable agreement with the  $\chi_{\ell m}^2$  distribution; one should take the autocorrelation structure of the error process into consideration for fMRI data analysis.

In Figure 2, we plot  $\widehat{r}_e(g)$  versus  $g$  in the left panels and  $\widehat{r}(g)$  versus  $g$  in the right panels for the MA(4) error process with  $\text{SNR} = 1$  based on a single

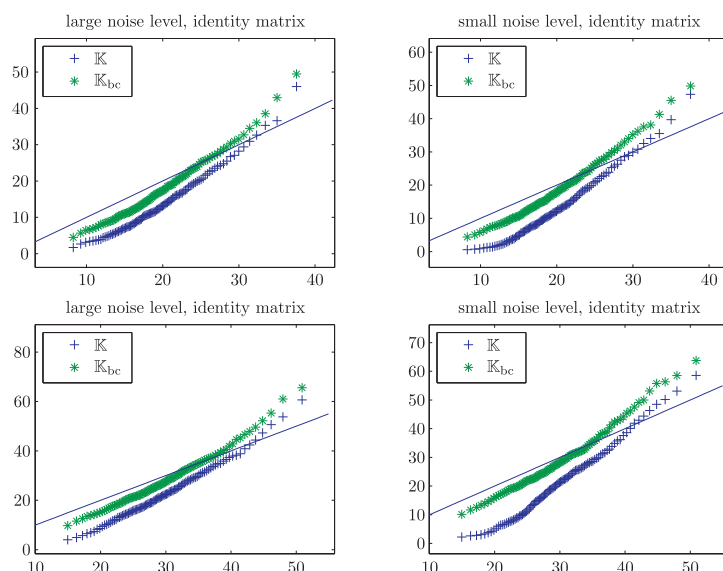


Figure 1. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the MA(4) error process, statistics calculated with  $\mathbf{R}$  estimated by  $\mathbf{I}_n$  and data-driven method for bandwidth choice. Statistics in the top panels calculated for a single event type, those in the bottom panels for two event types. The solid line is the 45 degree straight line.

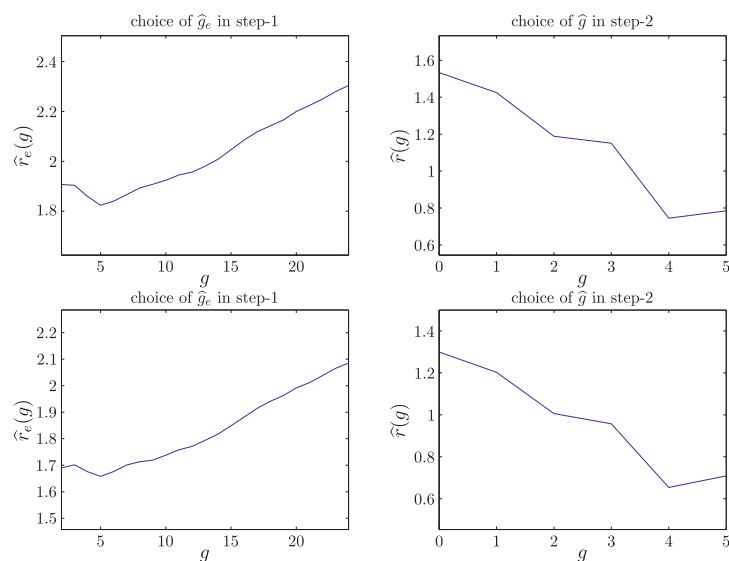


Figure 2. Plots of  $\hat{r}_e(g)$  versus  $g$  (left panels) and  $\hat{r}(g)$  versus  $g$  (right panels) for the MA(4) error process with SNR = 1 based on one realization. The top panels are for a single event type and the bottom panels correspond to two event types.

realization. The top panels correspond to a single type of stimulus, and the bottom panels are for two event types. Each plot shows a “V” shape with a global minimizer, which indicates that the data-driven method for choosing  $g_n$  is valid.

Figure 3 gives QQ plots of the percentiles of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  against those of the  $\chi_{\ell m}^2$  distribution for a single type of stimulus; Figure 4 shows QQ plots constructed for two event types. In each figure, statistics calculated with the true inverse autocorrelation matrix  $\mathbf{R}^{-1}$  and the theoretical optimal choice of bandwidth (see, e.g., (3) in Zhang (2003)) are plotted in the top panels, while those using  $\widehat{\mathbf{R}}_*^{-1}$  and data-driven method for the bandwidth choice are plotted in the bottom panels. The left and right panels correspond to large (SNR = 1) and small (SNR = 8) noise levels, respectively.

In the top panels of Figure 3, the finite sampling distributions of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  show reasonable agreement with the  $\chi_{\ell m}^2$  distribution for both large and small noise levels. For the plots in the bottom panels, we draw the same conclusion; the empirical distributions of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  calculated with  $\mathbf{R}^{-1}$  and  $\widehat{\mathbf{R}}_*^{-1}$  are well approximated by the  $\chi_{\ell m}^2$  distribution. In Figure 4, the QQ plots are constructed for two event types and the results are similar in spirit to the ones in Figure 3.

### 5.2. Example 2: ARMA(1, 3) error process

The error process was  $\epsilon(t_i) = \eta\epsilon(t_{i-1}) + z(t_i) + \xi_1 z(t_{i-1}) + \xi_2 z(t_{i-2}) + \xi_3 z(t_{i-3})$ , where  $\eta = 0.1$ ,  $\xi_1 = 0.9$ ,  $\xi_2 = 0.7$ ,  $\xi_3 = 0.25$  and  $\{z(t_i)\}$  were independent  $N(0, \sigma_z^2)$  and  $N(0, \sigma_z^2/8)$ . We took  $\sigma_z^2 = 0.4079^2$  for  $\ell = 1$ , and  $\sigma_z^2 = 0.3899^2$  for  $\ell = 2$ , such that the corresponding SNRs are about 1 and 8.

To check the validity of the data-driven method for choosing  $D$ , we plot  $\widehat{r}_*(D)$  versus  $D$  for the ARMA(1, 3) error process with SNR = 1 based on a single realization, in Figure 5. The left panel is for a single event type and the right panel for two event types. Each plot is approximately “V”-shaped, which indicates that it is a reasonable way to select  $\widehat{D}$  by minimizing  $\widehat{r}_*(D)$ .

The QQ plots for  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  against the  $\chi_{\ell m}^2$  distribution are presented in Figure 6 for one event type and in Figure 7 for two event types. All plots show that both  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  have an approximated  $\chi_{\ell m}^2$  distribution. Based on the conceivably good performance of  $\widehat{\mathbf{R}}_*^{-1}$  in the semiparametric test statistics, we can conclude that  $\widehat{\mathbf{R}}_*^{-1}$  serves as a satisfactory estimate of  $\mathbf{R}^{-1}$ .

### 5.3. Example 3: Sum of AR(1) and white noise error process

The error process consists of an AR(1) time series and a sequence of white noises,  $\epsilon(t_i) = \zeta_1(t_i) + \zeta_2(t_i)$ . Here  $\{\zeta_1(t_i)\}$  was the white noise, independent  $N(0, \sigma_\zeta^2)$  and  $N(0, \sigma_\zeta^2/8)$ , respectively;  $\{\zeta_2(t_i)\}$  were generated from the AR(1) model  $\zeta_2(t_i) = \rho\zeta_2(t_{i-1}) + z(t_i)$ , where  $\rho = 0.638$  and  $z(t_i) \sim N(0, \sigma_z^2)$  and



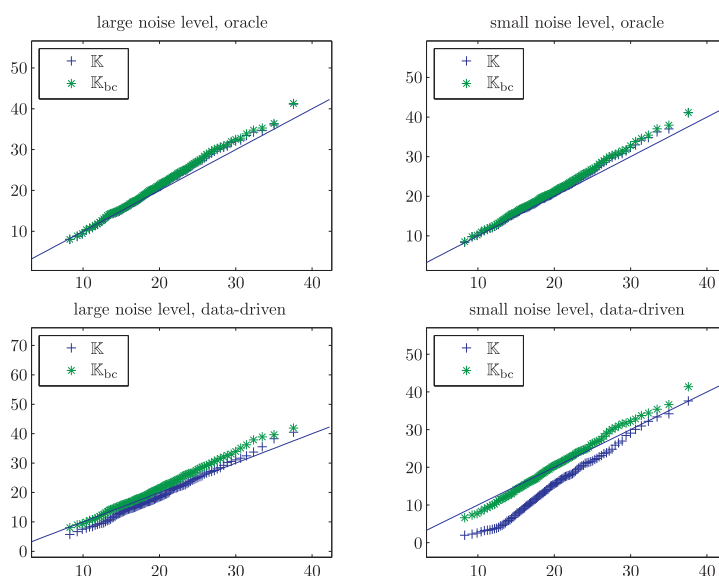


Figure 3. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the MA(4) error process with a single event type, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

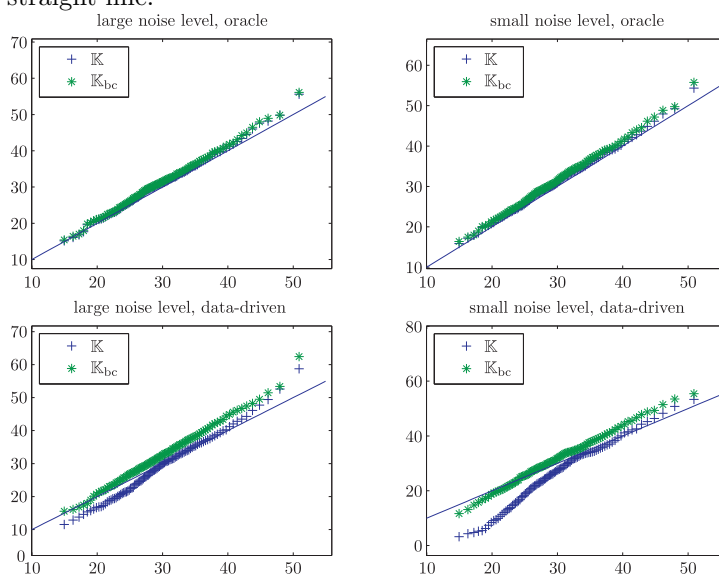


Figure 4. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the MA(4) error process with two event types, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

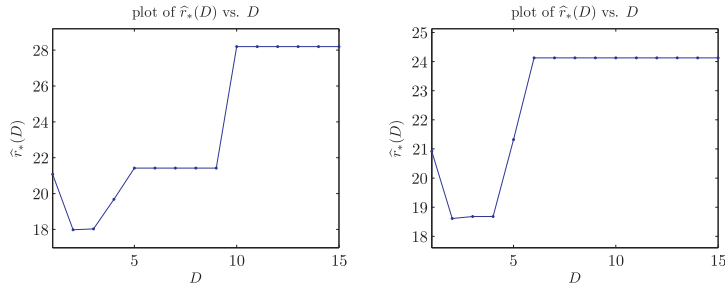


Figure 5. Plots of  $\hat{r}_*(D)$  versus  $D$  for the ARMA(1,3) error process with SNR = 1 based on a single realization. The left panel corresponds to a single event type and the right panel is for two event types. The points in each plot correspond to the ones used for the grid search method.

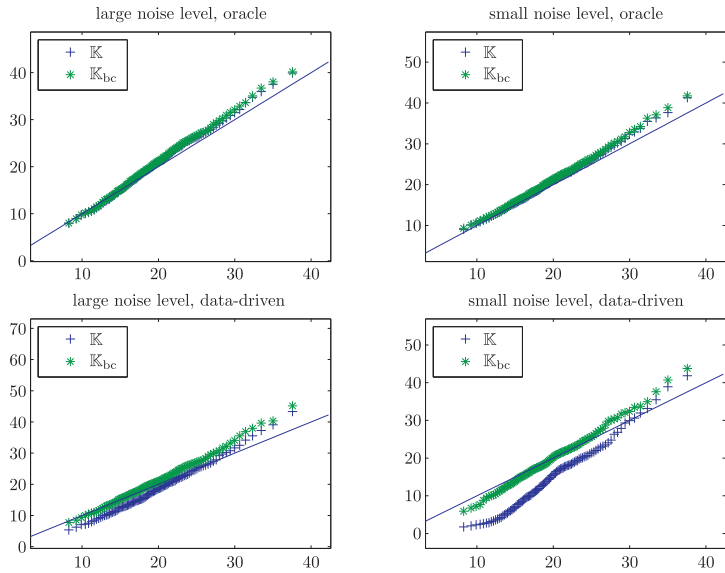


Figure 6. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the ARMA(1,3) error process with a single event type, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

$N(0, \sigma_z^2/8)$ , respectively. For a single event type, we took  $\sigma_\zeta^2 = 0.2430^2$  and  $\sigma_z^2 = 0.4861^2$ , and for two event types,  $\sigma_\zeta^2 = 0.2324^2$  and  $\sigma_z^2 = 0.4647^2$ , which give the SNRs of about 1 and 8.

Figure 8 and Figure 9 display the QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  versus the  $\chi_{\ell m}^2$  distribution. The former figure is constructed for a single event type while the latter one is made for two event types. The test statistics calculated by  $\mathbf{R}^{-1}$

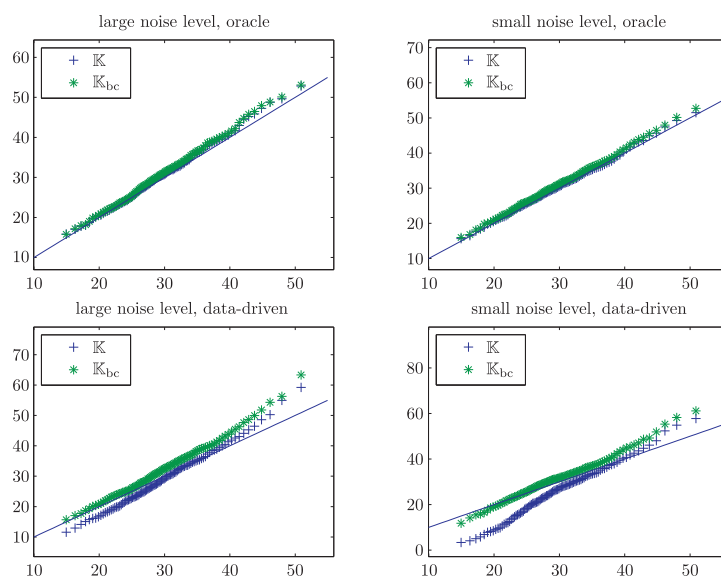


Figure 7. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the ARMA(1, 3) error process with two event types, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

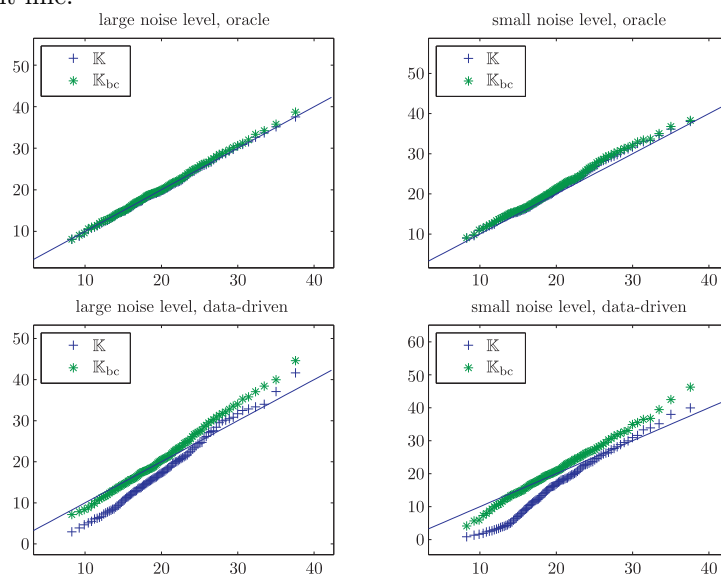


Figure 8. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the AR(1) + WN error process with a single event type, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

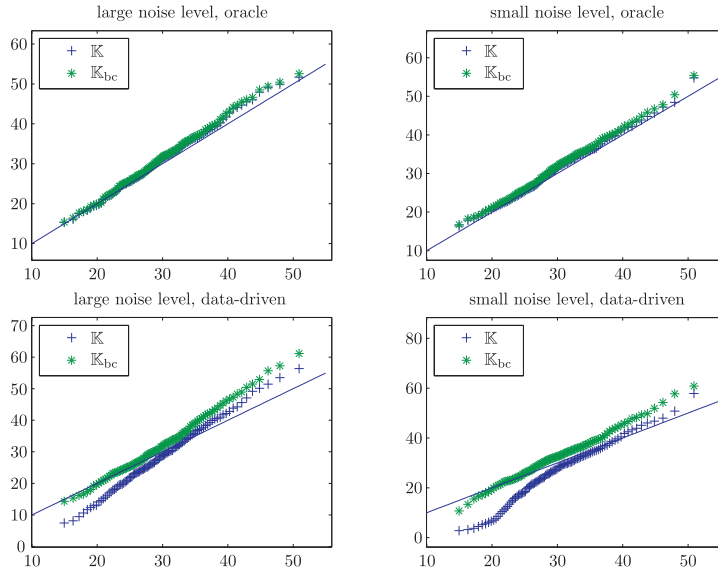


Figure 9. Empirical QQ plots of  $\mathbb{K}$  and  $\mathbb{K}_{bc}$  for the AR(1) + WN error process with two event types, statistics in the top panels calculated with  $\mathbf{R}^{-1}$  and theoretical optimal bandwidth choice, those in the bottom panels used  $\hat{\mathbf{R}}_*^{-1}$  and data-driven method for bandwidth choice. The solid line is the 45 degree straight line.

and  $\hat{\mathbf{R}}_*^{-1}$  have finite sampling distributions coinciding with the  $\chi_{\ell m}^2$  distribution fairly well.

### 5.4. Comparison of $\hat{\mathbf{R}}_*$ and $\hat{\mathbf{R}}_Z$

The difference between  $\hat{\mathbf{R}}_*$  and the estimator  $\hat{\mathbf{R}}_Z$  in Zhang et al. (2008) and Zhang and Yu (2008), is that we develop data-driven methods to select the banding parameter  $g_n$  and choose  $\omega$  and  $D$  for refining the estimator. The choice of the banding parameter for  $\hat{\mathbf{R}}_Z$  is  $g_n = 2$ , and hence  $\hat{\mathbf{R}}_Z$  is computationally faster than  $\hat{\mathbf{R}}_*$ . In the following, we compare the two estimators based on the error processes of Sections 5.1–5.3. We compare the average selected banding parameter by the data-driven method with  $g_n = 2$  used in Zhang et al. (2008) and Zhang and Yu (2008). The average loss of  $\hat{\mathbf{R}}_*$  and  $\hat{\mathbf{R}}_Z$  under the  $L_\infty$  norm,  $\|\hat{\mathbf{R}}_* - \mathbf{R}\|_\infty$  and  $\|\hat{\mathbf{R}}_Z - \mathbf{R}\|_\infty$ , were also compared.

Table 1 presents  $g_0$  (minimizer of the empirical version of (4.2)), the estimated banding parameter by the data-driven method  $\hat{g}$ , and the losses of  $\hat{\mathbf{R}}_*$  and  $\hat{\mathbf{R}}_Z$  under the  $L_\infty$  norm. From Table 1, we see that the data-driven method picks the banding parameter close to  $g_0$ . For the MA(4) and ARMA(1, 3) models, the loss of  $\hat{\mathbf{R}}_*$  is smaller than that of  $\hat{\mathbf{R}}_Z$ . For the AR(1) + WN model, where  $g_0 = 2$ ,

Table 1. Comparison of  $g_0$  and estimated banding parameter, losses of  $\hat{\mathbf{R}}_*$  and  $\hat{\mathbf{R}}_Z$ .

| SNR               | Model      | Banding parameter |                | Losses                    |                           |
|-------------------|------------|-------------------|----------------|---------------------------|---------------------------|
|                   |            | $g_0$             | $\hat{g}$ (SD) | $\hat{\mathbf{R}}_*$ (SD) | $\hat{\mathbf{R}}_Z$ (SD) |
| Single event type |            |                   |                |                           |                           |
| 1                 | MA(4)      | 4                 | 3.9 (0.04)     | 0.53 (0.03)               | 1.70 (0.01)               |
|                   | ARMA(1, 3) | 3                 | 2.8 (0.02)     | 0.34 (0.02)               | 1.09 (0.03)               |
|                   | AR(1) + WN | 2                 | 1.6 (0.03)     | 2.23 (0.02)               | 2.14 (0.01)               |
| 8                 | MA(4)      | 4                 | 3.9 (0.04)     | 0.54 (0.03)               | 1.69 (0.01)               |
|                   | ARMA(1, 3) | 3                 | 2.7 (0.02)     | 0.38 (0.02)               | 1.04 (0.03)               |
|                   | AR(1) + WN | 2                 | 1.6 (0.03)     | 2.25 (0.02)               | 2.14 (0.01)               |
| Two event types   |            |                   |                |                           |                           |
| 1                 | MA(4)      | 4                 | 3.9 (0.04)     | 0.52 (0.03)               | 1.70 (0.01)               |
|                   | ARMA(1, 3) | 3                 | 2.8 (0.02)     | 0.36 (0.02)               | 1.11 (0.03)               |
|                   | AR(1) + WN | 2                 | 1.6 (0.03)     | 2.24 (0.02)               | 2.14 (0.01)               |
| 8                 | MA(4)      | 4                 | 3.9 (0.04)     | 0.53 (0.03)               | 1.70 (0.01)               |
|                   | ARMA(1, 3) | 3                 | 2.8 (0.02)     | 0.38 (0.02)               | 1.05 (0.03)               |
|                   | AR(1) + WN | 2                 | 1.7 (0.03)     | 2.24 (0.02)               | 2.14 (0.01)               |

$\hat{\mathbf{R}}_Z$  performs slightly better since the choice of the banding parameter is exactly  $g_n = 2$ .

### 6. Data Analysis

In an emotional control study, subjects saw a series of negative or positive emotional images, and were asked to either suppress or enhance their emotional responses to the image, or to simply attend to the image. The sequence of trials was randomized. The time between successive trials also varied. The size of the whole brain dataset is  $64 \times 64 \times 30$ . At each voxel, the time series has 6 runs, each containing 185 observations with a time resolution of 2 secs. Refer to Zhang and Yu (2008) for details of the dataset and detection of activated brain regions. The study aims to estimate the BOLD (Blood Oxygenation Level-Dependent) response to each of the trial types for 1–18 seconds following the image onset. The length of the estimated HRF is set equal to 18.

The methodological difference between Zhang and Yu (2008) and this paper is between  $\hat{\mathbf{R}}_Z^{-1}$  and  $\hat{\mathbf{R}}_*^{-1}$  for estimating  $\mathbf{R}^{-1}$ , where  $\hat{\mathbf{R}}_*^{-1}$  selects  $g_n$  and  $D$  via data-driven methods. Our numerical results indicate that the banding parameter  $\hat{g}$  selected for  $\hat{\mathbf{R}}_*^{-1}$  is very close to  $g_n = 2$  used for  $\hat{\mathbf{R}}_Z^{-1}$ . They in turn yield nearly identical estimates of HRF and the corresponding test statistics, without altering the declared significance. For example, Table 2 reports the banding parameter and  $D$  selected for  $\hat{\mathbf{R}}_*^{-1}$  at two activated voxels. Figure 10 compares the estimates of HRF using the semi-parametric model based on  $\hat{\mathbf{R}}_*^{-1}$  and  $\hat{\mathbf{R}}_Z^{-1}$ ,

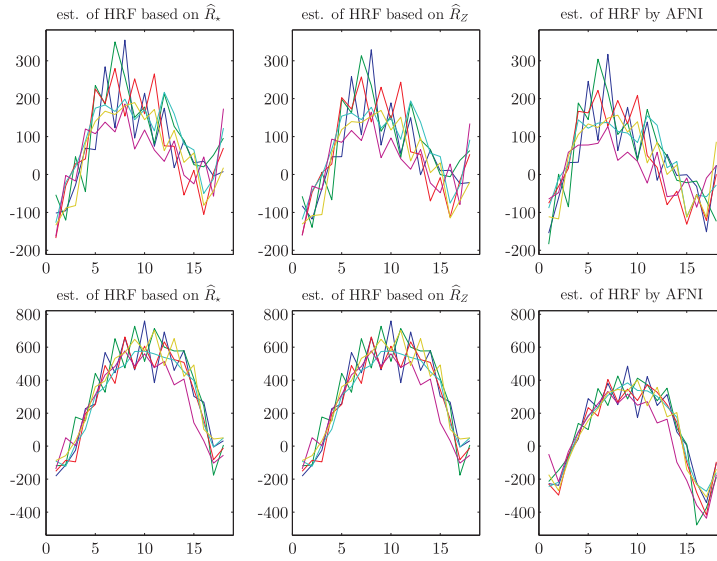


Figure 10. Compare estimates of HRF for each of six stimuli. Top panels: at voxel (24, 32, 7); bottom panels: at voxel (49, 41, 10).

Table 2. Compare banding parameters used for  $\widehat{\mathbf{R}}_*^{-1}$  and  $\widehat{\mathbf{R}}_Z^{-1}$  at two voxels.

| voxel        | for $\widehat{\mathbf{R}}_*^{-1}$                      | for $\widehat{\mathbf{R}}_Z^{-1}$ |
|--------------|--|-----------------------------------|
| (24, 32, 7)  | $\widehat{g} = 1, \widehat{g}_e = 2, \widehat{D} = 21$ | $g_n = 2$                         |
| (49, 41, 10) | $\widehat{g} = 2, \widehat{g}_e = 2, \widehat{D} = 16$ | $g_n = 2$                         |

and the parametric model via AFNI software (Zhang et al. (2008)). Similar comparison can be performed at other voxels.

## 7. Discussion

Dealing with the temporally correlated error process is an important issue in fMRI data analysis. The existing methods for handling this problem require strong assumptions on the error process. We consider estimating the error autocorrelation matrix and its inverse under a more general assumption on the error process, that allows for a wider range of the error correlation structure. The existing methods for estimating covariance matrix are not applicable to our case because of the special structure of the semiparametric model for fMRI data. We adopt a difference-based method, and develop a banded estimate of  $\mathbf{R}$  and a refined estimate of  $\mathbf{R}^{-1}$ . Compared with the smoothing-based method, the difference-based method enjoys good practical performance and computational efficiency. We demonstrate consistency of our proposed estimators and provide data-driven approaches for calculating the estimates in practice. Hence, we pro-

vide a computationally efficient and consistent estimate of  $\mathbf{R}^{-1}$  for semiparametric inference.

The optimal convergence rate of the covariance matrix estimate is an important topic, studied in Cai, Zhang, and Zhou (2010), Cai, Ren, and Zhou (2012), Cai and Zhou (2012), and Xiao and Wu (2012). These results are not applicable to our case. It would be interesting to investigate the optimal convergence rate of the estimates of  $\mathbf{R}$  and  $\mathbf{R}^{-1}$  under our setting. Our proposed estimator is particularly useful for analyzing fMRI data under the semiparametric model, but may not be easily used to estimate the error correlation matrix for other datasets, or for fMRI data with different models.

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**Estimation of the Error Autocorrelation  
Matrix in Semiparametric Model for fMRI Data**

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**Supplementary Material**

**Summary**

The supplementary material includes the Appendix of the paper. It includes

- I. the conditions for theoretical analysis,
- II. some notation used in the proofs,
- III. proofs of Lemmas 1–9,
- IV. proofs of the main theoretical results (Theorems 1 and 2, Propositions 1–4) in the paper.

## Appendix: Conditions and Proofs

In this part, we will give the conditions and proofs of the main results in this paper.

### Condition A.

- A1. In model (2.1),  $\epsilon(t_i) = \sum_{j=-\infty}^{\infty} \phi_{n;j} w_{i-j}$ , where  $\phi_{n;0} = 1$ ; for any  $n \geq 1$ , there exists  $1 \leq g_n \leq n-1$  and  $\alpha_n > 4$ , such that  $|\phi_{n;j}| \leq C$  for  $|j| \leq g_n/2$  and  $|\phi_{n;j}| \leq C|2j|^{-\alpha_n}$  for  $|j| > g_n/2$ , with a constant  $C > 0$ ;  $\{w_i\}$  is a sequence of independent white noises with  $E(w_i) = 0$ ,  $E(w_i^2) = \sigma_w^2$ , and  $\sup_i E(w_i^4) < \infty$ .
- A2. Suppose  $\lambda_{\min}(\Sigma_1) > C > 0$ , where  $\Sigma_1$  is an  $m\ell \times m\ell$  matrix consisting of  $\ell \times \ell$  blocks, i.e.,  $\Sigma_1 = (\Sigma_{i,j})_{i,j=1}^{\ell}$ .  $\Sigma_{i,j}$  is an  $m \times m$  matrix defined as  $\Sigma_{i,j}(u, v) = \text{cov}\{\mathbf{z}_{v,n-1}^T(\mathbf{D}_1 \mathbf{S}_i) \mathbf{z}_{1,m}, \mathbf{z}_{u,n-1}^T(\mathbf{D}_1 \mathbf{S}_j) \mathbf{z}_{1,m}\}$ , for  $1 \leq u, v \leq m$ ,  $1 \leq i, j \leq \ell$ , and  $\mathbf{z}_{p,q}$  is the  $p$ th column of the  $q \times q$  identity matrix.
- A3. The second derivative of the drift function  $d(t)$  is continuous and bounded, i.e.,  $|d''(t)| \leq C$ .
- A4. In model (2.1),  $\{s_i(\cdot)\}$ ,  $i = 1, \dots, \ell$ , are independent of  $\{\epsilon(\cdot)\}$ . For each  $1 \leq i \leq \ell$ ,  $\{s_i(\cdot)\}$  is a stationary  $g_s$ -dependent time series, where  $g_s > 0$  is a fixed integer, and  $E\{s_i^4(t)\} \leq C < \infty$ . Furthermore,  $\{s_1(t_u), \dots, s_{\ell}(t_u)\}$  and  $\{s_1(t_v), \dots, s_{\ell}(t_v)\}$  are independent if  $|u - v| > g_s$ . When  $|u - v| \leq g_s$ ,  $E\{s_i(t_u) s_j(t_v)\}$  depends on  $u$  and  $v$  only through  $u - v$ , for any  $1 \leq i, j \leq \ell$ .
- A5. Suppose  $t_i = i/n$ ,  $i = 1, \dots, n$ .
- A6. Assume  $0 < c \leq \lambda_{\min}(\mathbf{R}) \leq \lambda_{\max}(\mathbf{R}) \leq C$ , where  $c$  and  $C$  are constants.

### Condition B.

- B1. In model (2.1),  $\epsilon$  is a stationary  $g_n$ -dependent process with  $E\{\epsilon(t_i)\} = 0$ ,  $c \leq \text{var}\{\epsilon(t_i)\} \leq C g_n$  and  $[E\{\epsilon(t_i)^4\}]^{1/2} = O(\gamma_e(0))$ , where  $1 \leq g_n \leq n-1$ .

**Notation.** Now, we will give some notation that will be used in the proofs.

1. Define  $\epsilon_1 = \mathbf{D}_1 \mathbf{d} + \mathbf{D}_1 \epsilon$ . Then,  $\epsilon_1(t_i) = \epsilon(t_i) - \epsilon(t_{i-1}) + d(t_i) - d(t_{i-1})$ , where  $\epsilon_1(t_i) = \mathbf{z}_{i-1, n-1}^T \epsilon_1$ , for  $i = 2, \dots, n$ .
2. Define  $\mathbf{e}_0 = \mathbf{D}_2 \epsilon$ ,  $\mathbf{d}_0 = \mathbf{D}_2 \mathbf{d}$  and  $\delta = \mathbf{D}_2 \mathbf{S}(\mathbf{h} - \hat{\mathbf{h}}_{\text{DBE}})$ . Then,  $\mathbf{e} = \mathbf{e}_0 + \mathbf{d}_0$  and  $\hat{\mathbf{e}} = \mathbf{e} + \delta$ . Also,  $e_0(t_i) = \epsilon(t_i) - 2\epsilon(t_{i-1}) + \epsilon(t_{i-2})$ ,  $d_0(t_i) = d(t_i) - 2d(t_{i-1}) + d(t_{i-2})$  and  $e(t_i) = e_0(t_i) + d_0(t_i)$ , where  $e_0(t_i) = \mathbf{z}_{i-2, n-2}^T \mathbf{e}_0$ ,  $d_0(t_i) = \mathbf{z}_{i-2, n-2}^T \mathbf{d}_0$  and  $\delta(t_i) = \mathbf{z}_{i-2, n-2}^T \delta$ , for  $i = 3, \dots, n$ .
3. For a matrix  $Z$ , denote by  $Z(i, j)$  the entry of  $Z$  in the  $i$ th row and  $j$ th column.

**Proof.** We will present and prove Lemmas 1–9, which will be needed in the proofs of the main results.

**Lemma 1** For the  $(k+1) \times (k+1)$  matrix  $A_k$  and  $(k+1) \times 2$  matrix  $B_k$  in (2.5) with  $k \geq 1$ ,

$$\begin{aligned} \text{(i)} \quad & \|A_k^{-1}\|_1 \leq Ck^4, \\ \text{(ii)} \quad & \|\mathbf{z}_{1,k+1}^T A_k^{-1}\|_\infty \leq Ck^3, \\ \text{(iii)} \quad & \|A_k^{-1} B_k\|_1 \leq Ck^2, \quad \|\mathbf{z}_{1,k+1}^T A_k^{-1} B_k\|_\infty \leq Ck, \end{aligned}$$

where  $\mathbf{z}_{p,q}$  is the  $p$ th column of the  $q \times q$  identity matrix.

*Proof:* Let  $G$ ,  $E$ ,  $H$  and  $K$  be  $(k+1) \times (k+1)$  matrices defined as follows:

$$G = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}^{(k+1) \times (k+1)},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then,  $A_k = K(G + E + H)$ . To prove part (i), it suffices to show that  $G + E + H$  is positive definite for any  $k \geq 1$ , and  $\|(G + E + H)^{-1}\|_1 = O(k^4)$ , since  $\|A_k^{-1}\|_1 \leq \|(G + E + H)^{-1}\|_1 \|K^{-1}\|_1 = \|(G + E + H)^{-1}\|_1$ .

First, we will show that  $G + E + H$  is positive definite for any  $k \geq 2$ , as the result is obvious for  $k = 1$ . From Theorem 2 of Hoskins and Ponzio (1972),  $G$  is positive definite. Since  $E$  is positive semidefinite,  $G + E$  is positive definite. By the particular form of matrix  $H$ ,  $\det(G + E + H) > 0$  is a necessary and sufficient condition for  $G + E + H$  to be positive definite. We can express  $G + E + H$  as a block matrix in the following way:

$$G + E + H = \begin{pmatrix} J_1 & J_2 \\ J_2^T & J_3 \end{pmatrix},$$

where

$$J_1 = \begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -4 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2 \times (k-1)},$$

$$J_3 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}_{(k-1) \times (k-1)}.$$

Then,

$$G + E + H = \begin{pmatrix} \mathbf{I}_{2 \times 2} & J_2 \\ \mathbf{0}_{(k-1) \times 2} & J_3 \end{pmatrix} \begin{pmatrix} J_1 - J_2 J_3^{-1} J_2^T & \mathbf{0}_{2 \times (k-1)} \\ J_3^{-1} J_2^T & \mathbf{I}_{(k-1) \times (k-1)} \end{pmatrix},$$

so  $\det(G + E + H) = \det(J_3) \det(J_1 - J_2 J_3^{-1} J_2^T)$ . Due to Theorem 2 in Hoskins and Ponzo (1972),  $\det(J_3) > 0$ . Now, we only need to show  $\det(J_1 - J_2 J_3^{-1} J_2^T) > 0$ .

Let  $x_{i,j} = \mathbf{z}_{i,k-1}^T J_3^{-1} \mathbf{z}_{j,k-1}$ . From Theorem 5 of Hoskins and Ponzo (1972),  $x_{1,1} = (k-1)k/\{(k+1)(k+2)\}$ ,  $x_{2,1} = x_{1,2} = 2(k-1)(k-2)/\{(k+1)(k+2)\}$  and  $x_{2,2} = (k-1)(k-2)(5k-6)/\{k(k+1)(k+2)\}$ . By direct calculation,

$$J_1 - J_2 J_3^{-1} J_2^T = \begin{pmatrix} 3 - (k-1)k/\{(k+1)(k+2)\} & -4 + 2(k-1)/(k+1) \\ -4 + 2(k-1)/(k+1) & 7 - (5k+6)(k-1)/\{k(k+1)\} \end{pmatrix}.$$

Thus,  $\det(J_1 - J_2 J_3^{-1} J_2^T) = 12(2k+3)/\{k(k+1)^2(k+2)\} > 0$ , which implies that  $\det(G + E + H) > 0$  and hence  $G + E + H$  is positive definite.

Next, we will show  $\|(G + E + H)^{-1}\|_1 = O(k^4)$  as  $k \rightarrow \infty$ .

Since the ranks of  $E$  and  $H$  are both 1, from Miller (1981),  $(G + E)^{-1} = G^{-1} - \nu_1 G^{-1} E G^{-1}$  and  $(G + E + H)^{-1} = (G + E)^{-1} - \nu_2 (G + E)^{-1} H (G + E)^{-1}$ , where  $\nu_1 = \{1 + \text{tr}(G^{-1} E)\}^{-1}$  and  $\nu_2 = [1 + \text{tr}\{(G + E)^{-1} H\}]^{-1}$ . Therefore,

$$\mathbf{z}_{i,k+1}^T (G + E)^{-1} \mathbf{z}_{j,k+1} = a_{i,j} - \nu_1 a_{i,2} a_{2,j}, \quad (\text{S.1})$$

$$\begin{aligned} \mathbf{z}_{i,k+1}^T (G + E + H)^{-1} \mathbf{z}_{j,k+1} &= (a_{i,j} - \nu_1 a_{i,2} a_{2,j}) \\ &\quad + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,j} - \nu_1 a_{1,2} a_{2,j}), \end{aligned} \quad (\text{S.2})$$

where  $a_{i,j} = \mathbf{z}_{i,k+1}^T G^{-1} \mathbf{z}_{j,k+1}$ , for  $1 \leq i, j \leq k+1$ .

From Theorem 5 of Hoskins and Ponzo (1972), for  $i = 1, \dots, k+1$ ,

$$a_{i,1} = \frac{i(k+2-i)(k+3-i)}{(k+3)(k+4)}, \quad (\text{S.3})$$

$$a_{i,2} = \frac{(k+2-i)(k+3-i)\{i(3k+4) - (k+4)\}}{(k+2)(k+3)(k+4)}. \quad (\text{S.4})$$

Direct calculation leads to

$$\nu_1 = \{1 + \text{tr}(G^{-1} E)\}^{-1} = (1 + a_{2,2})^{-1} \xrightarrow{k \rightarrow \infty} 1/6, \quad (\text{S.5})$$

$$\begin{aligned} \nu_2 &= [1 + \text{tr}\{(G + E)^{-1} H\}]^{-1} = [1 - 3\{a_{1,1} - a_{1,2}^2/(1 + a_{2,2})\}]^{-1} \\ &= O(k^3/4). \end{aligned} \quad (\text{S.6})$$

From (S.3) and (S.4), for any  $i = 1, \dots, k+1$ , we have  $0 < a_{i,1}, a_{i,2} = O(k)$  and

$$\begin{aligned} a_{i,1} - \nu_1 a_{i,2} a_{2,1} &= \frac{(k+2-i)(k+3-i)}{(k+3)(k+4)} \frac{4k^2i + 22ki + 24i + 2k^3 + 10k^2 + 8k}{6k^3 + 18k^2 + 30k + 24} \\ &< 2. \end{aligned} \quad (\text{S.7})$$

Since  $G^{-1}$  is symmetric, we can immediately get  $0 < a_{1,j} - \nu_1 a_{1,2} a_{2,j} < 2$  for any  $j = 1, \dots, k+1$ .

By Theorem 3 of Hoskins and Ponzo (1972),

$$\begin{aligned} \sum_{i=1}^{k+1} a_{i,1} &= \sum_{i=1}^{k+1} |a_{i,1}| = \frac{(k+1)(k+2)}{12}, \\ \sum_{i=1}^{k+1} a_{i,2} &= \sum_{i=1}^{k+1} |a_{i,2}| = \frac{k(k+1)}{4}. \end{aligned} \quad (\text{S.8})$$

Theorem 4 of Hoskins and Ponzo (1972) indicates that  $\|G^{-1}\|_1 = O(k^4)$ , which together with (S.1)–(S.8) implies,

$$\begin{aligned} &\|(G + E + H)^{-1}\|_1 \\ &= \max_{1 \leq j \leq k+1} \sum_{i=1}^{k+1} |(a_{i,j} - \nu_1 a_{i,2} a_{2,j}) + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,j} - \nu_1 a_{1,2} a_{2,j})| \\ &\leq \|G^{-1}\|_1 + \nu_1 \max_{1 \leq j \leq k+1} |a_{2,j}| \sum_{i=1}^{k+1} |a_{i,2}| \\ &\quad + 3\nu_2 \max_{1 \leq j \leq k+1} |a_{1,j} - \nu_1 a_{1,2} a_{2,j}| \sum_{i=1}^{k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \\ &= O(k^4) + O(1)O(k)O(k^2) + O(k^3)O(1)O(k) = O(k^4). \end{aligned}$$

For part (ii), by (S.1)–(S.8),

$$\begin{aligned} &\|\mathbf{z}_{1,k+1}^T A_k^{-1}\|_\infty \leq \|\mathbf{z}_{1,k+1}^T (G + E + H)^{-1}\|_\infty \|K^{-1}\|_\infty = \|(G + E + H)^{-1} \mathbf{z}_{1,k+1}\|_\infty \\ &= \max_{1 \leq i \leq k+1} |(a_{i,1} - \nu_1 a_{i,2} a_{2,1}) + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,1} - \nu_1 a_{1,2} a_{2,1})| \\ &\leq \max_{1 \leq i \leq k+1} |a_{i,1}| + \nu_1 |a_{2,1}| \max_{1 \leq i \leq k+1} |a_{i,2}| + 3\nu_2 |a_{1,1} - \nu_1 a_{1,2} a_{2,1}| \max_{1 \leq i \leq k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \\ &= O(k) + O(1)O(1)O(k) + O(k^3)O(1)O(1) = O(k^3). \end{aligned}$$

For part (iii), we only consider  $k > 3$ , as the result for  $k \leq 3$  is obvious.

Since  $A_k^{-1} \mathbf{z}_{k+1,k+1} = (G + E + H)^{-1} K^{-1} \mathbf{z}_{k+1,k+1} = (G + E + H)^{-1} \mathbf{z}_{k+1,k+1}$ , from (S.2),

$$\begin{aligned} &\|A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 = \|(G + E + H)^{-1} \mathbf{z}_{k+1,k+1}\|_1 \\ &\leq \sum_{i=1}^{k+1} |a_{i,k+1}| + \nu_1 |a_{2,k+1}| \sum_{i=1}^{k+1} |a_{i,2}| + 3\nu_2 |a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1}| \sum_{i=1}^{k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \end{aligned}$$

$$\equiv \text{I} + \text{II} + \text{III}. \quad (\text{S.9})$$

Theorem 3 in Hoskins and Ponzo (1972) implies that  $\text{I} = O(k^2)$ .

From Theorem 5 in Hoskins and Ponzo (1972), for  $i = 1, \dots, k+1$ ,

$$a_{i,k+1} = i(i+1)(k-i+2)/\{(k+3)(k+4)\}, \quad (\text{S.10})$$

which together with (S.5) and (S.8) implies that  $\text{II} = O(1)O(k^{-1})O(k^2) = O(k)$ .

By (S.3), (S.4), (S.5) and (S.10),

$$a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1} = \frac{2(k+1)}{(k+3)(k+4)} - \frac{a_{1,2}}{1+a_{2,2}} \frac{6k}{(k+3)(k+4)} = O(k^{-2}). \quad (\text{S.11})$$

From (S.6), (S.7) and (S.11),  $\text{III} = O(k^3)O(k^{-2})O(k) = O(k^2)$ . Therefore, by (S.9),  $\|A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 = O(k^2)$ . Similar arguments reveal that  $\|A_k^{-1} \mathbf{z}_{k,k+1}\|_1 = O(k^2)$ . Since, for  $k > 3$ ,  $B_k = [\mathbf{z}_{k,k+1} - 4\mathbf{z}_{k+1,k+1} \quad \mathbf{z}_{k+1,k+1}]$ ,

$$\begin{aligned} \|A_k^{-1} B_k\|_1 &\leq \|A_k^{-1} \mathbf{z}_{k,k+1} - 4A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 + \|A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 \\ &\leq \|A_k^{-1} \mathbf{z}_{k,k+1}\|_1 + 4\|A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 + \|A_k^{-1} \mathbf{z}_{k+1,k+1}\|_1 = O(k^2). \end{aligned}$$

From (S.2), (S.7) and (S.11),

$$\begin{aligned} &|\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k+1,k+1}| \\ &= |(a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1}) + 3\nu_2(a_{1,1} - \nu_1 a_{1,2} a_{2,1})(a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1})| \\ &= O(k^{-2}) + O(k^3)O(k^{-2}) = O(k). \end{aligned}$$

Similarly, we can show  $|\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k,k+1}| = O(k)$ . Therefore,

$$\begin{aligned} \|\mathbf{z}_{1,k+1}^T A_k^{-1} B_k\|_\infty &\leq |\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k,k+1} - 4\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k+1,k+1}| + |\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k+1,k+1}| \\ &\leq |\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k,k+1}| + 4|\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k+1,k+1}| + |\mathbf{z}_{1,k+1}^T A_k^{-1} \mathbf{z}_{k+1,k+1}| = O(k). \end{aligned}$$

Now we complete the proof. ■

**Lemma 2** Under Condition A1 (or B1), for any  $k \in \{0, 1, \dots, g_n\}$  and  $\tau_{0;n} > 0$  that satisfies  $\tau_{0;n}^{-1} = o(n/g_n^2)$  as  $n \rightarrow \infty$ ,

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i) e_0(t_{i+k})\} - \gamma_e(k)\right| \geq \tau_{0;n}\right) \leq C \frac{g_n^3}{n\tau_{0;n}^2},$$

where  $e_0(t_i) = \epsilon(t_i) - 2\epsilon(t_{i-1}) + \epsilon(t_{i-2})$  as in Notation 2.

*Proof:* First, we will give the proof under Condition A1. Since  $\epsilon(t_i) = \sum_{j=-\infty}^{\infty} \phi_{n;j} w_{i-j}$ , we have  $e_0(t_i) = \sum_{j=-\infty}^{\infty} \psi_j w_{i-j}$ , where  $\psi_j \equiv \psi_{n;j} = \phi_{n;j} - 2\phi_{n;j-1} + \phi_{n;j-2}$ . Define  $\tilde{e}_0(t_i) = \sum_{j=-g_n+2}^{g_n+2} \psi_j w_{i-j}$  and  $\tilde{\gamma}_e(k) = \text{cov}\{\tilde{e}_0(t_i), \tilde{e}_0(t_{i+k})\}$ . Then,

$$\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i) e_0(t_{i+k})\} - \gamma_e(k)\right| \leq \frac{1}{n} \left| \sum_{i=3}^{n-k} \{e_0(t_i) e_0(t_{i+k}) - \tilde{\gamma}_e(k)\} \right| + (k+2)|\gamma_e(k)|/n$$

$$\begin{aligned}
&\leq \frac{1}{n} \left| \sum_{i=3}^{n-k} \{e_0(t_i)e_0(t_{i+k}) - \gamma_e(k) - \tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) + \tilde{\gamma}_e(k)\} \right| \\
&\quad + \frac{1}{n} \left| \sum_{i=3}^{n-k} \{\tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) - \tilde{\gamma}_e(k)\} \right| + (k+2)|\gamma_e(k)|/n \\
&\equiv \text{I} + \text{II} + \text{III}. \tag{S.12}
\end{aligned}$$

Since  $\tau_{0;n}^{-1} = o(n/g_n^2)$  as  $n \rightarrow \infty$  and  $|\gamma_e(k)| \leq \gamma_e(0) = O(g_n)$ , there exists a constant  $L_0$ , such that, for any  $n > L_0$  and  $k \in \{0, 1, \dots, g_n\}$ ,  $\text{III} < \tau_{0;n}/3$ . From (S.12), for  $n > L_0$ ,

$$\mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\right| \geq \tau_{0;n}\right) \leq \mathbb{P}(\text{I} \geq \tau_{0;n}/3) + \mathbb{P}(\text{II} \geq \tau_{0;n}/3). \tag{S.13}$$

From Lemma 3,

$$\mathbb{P}(\text{II} \geq \tau_{0;n}/3) = O(g_n^3/(n\tau_{0;n}^2)). \tag{S.14}$$

Therefore, we only need to consider term I. By Markov inequality,

$$\begin{aligned}
&\mathbb{P}(\text{I} \geq \tau_{0;n}/3) \\
&\leq \left(\frac{3}{n\tau_{0;n}}\right)^2 E\left[\left|\sum_{i=3}^{n-k} \{e_0(t_i)e_0(t_{i+k}) - \gamma_e(k) - \tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) + \tilde{\gamma}_e(k)\}\right|^2\right]. \tag{S.15}
\end{aligned}$$

For any fixed  $n > L_0$  and  $k \in \{0, 1, \dots, g_n\}$ , let  $Q_i = e_0(t_i)e_0(t_{i+k}) - \tilde{e}_0(t_i)\tilde{e}_0(t_{i+k})$ . Define  $D_{i,u} = E(Q_i | w_{i-u}, w_{i-u+1}, \dots) - E(Q_i | w_{i-u+1}, w_{i-u+2}, \dots)$ , for  $u = 0, \pm 1, \pm 2, \dots$ . It holds that  $\sum_{u=-\infty}^{\infty} D_{i,u} = Q_i - E(Q_i) = e_0(t_i)e_0(t_{i+k}) - \gamma_e(k) - \tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) + \tilde{\gamma}_e(k)$  almost surely. For any  $u$ ,  $\{D_{i,u}\}_{i=n-k}^3$  is a martingale difference sequence w.r.t.  $\mathcal{F}_{i,u} = \sigma\{w_{i-u}, w_{i-u+1}, \dots\}$ , i.e.,  $E(D_{i,u} | \mathcal{F}_{i+1,u}) = 0$  for  $i = n-k, \dots, 3$ . In the following, define  $\psi_j^* = \psi_j$  for  $|j| \leq g_n + 2$  and  $\psi_j^* = 0$  for  $|j| > g_n + 2$ . Direct calculation leads to,

$$\begin{aligned}
&E(D_{i,u}^2) \\
&= E\left\{\left(\psi_{u+k}w_{i-u} \sum_{j=-\infty}^{u-1} \psi_j w_{i-j} + \psi_u w_{i-u} \sum_{j=-\infty}^{k+u-1} \psi_j w_{i+k-j} + \psi_{u+k}\psi_u w_{i-u}^2 - \psi_{u+k}\psi_u \sigma_w^2 \right. \right. \\
&\quad \left. \left. - \psi_{u+k}^* w_{i-u} \sum_{j=-\infty}^{u-1} \psi_j^* w_{i-j} - \psi_u^* w_{i-u} \sum_{j=-\infty}^{k+u-1} \psi_j^* w_{i+k-j} - \psi_{u+k}^* \psi_u^* w_{i-u}^2 + \psi_{u+k}^* \psi_u^* \sigma_w^2\right)^2\right\} \\
&= O(g_n(\psi_u^2 + \psi_{u+k}^2)),
\end{aligned}$$

which together with Lemma 7 of Xiao and Wu (2012) implies

$$E\left\{\left(\sum_{i=3}^{n-k} D_{i,u}\right)^2\right\} \leq \sum_{i=3}^{n-k} E(D_{i,u}^2) = O(n g_n (\psi_u^2 + \psi_{u+k}^2)). \tag{S.16}$$

By Minkowski inequality and (S.16), for the expectation term in (S.15),

$$E\left[\left|\sum_{i=3}^{n-k} \{e_0(t_i)e_0(t_{i+k}) - \gamma_e(k) - \tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) + \tilde{\gamma}_e(k)\}\right|^2\right]$$

$$= E \left[ \left| \sum_{i=3}^{n-k} \sum_{u=-\infty}^{\infty} D_{i,u} \right|^2 \right] \leq \left[ \sum_{u=-\infty}^{\infty} \left\{ E \left( \left| \sum_{i=3}^{n-k} D_{i,u} \right|^2 \right) \right\}^{1/2} \right]^2 = O(n g_n^3).$$

From (S.15),

$$P(I \geq \tau_{0;n}/3) = O(g_n^3/(n\tau_{0;n}^2)). \quad (\text{S.17})$$

Combining (S.13), (S.14) and (S.17), we finish the proof for  $n > L_0$ . It is easy to see that the result is true for  $n \leq L_0$ .

Next, we will prove the Lemma under Condition B1. For any  $k = 0, 1, \dots, g_n$ ,

$$\begin{aligned} & P \left( \left| \frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k) \right| \geq \tau_{0;n} \right) \\ = & P \left( \left| \frac{1}{4n} \left[ \sum_{i=3}^{n-k} \{e_0(t_i) + e_0(t_{i+k})\}^2 - \sum_{i=3}^{n-k} \{e_0(t_i) - e_0(t_{i+k})\}^2 \right] \right. \right. \\ & \left. \left. - \frac{1}{4} [\{2\gamma_e(0) + 2\gamma_e(k)\} - \{2\gamma_e(0) - 2\gamma_e(k)\}] \right| \geq \tau_{0;n} \right) \\ \leq & P \left( \left| \frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i) + e_0(t_{i+k})\}^2 - \{2\gamma_e(0) + 2\gamma_e(k)\} \right| \geq 2\tau_{0;n} \right) \\ & + P \left( \left| \frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i) - e_0(t_{i+k})\}^2 - \{2\gamma_e(0) - 2\gamma_e(k)\} \right| \geq 2\tau_{0;n} \right) \\ \leq & P \left( \left| \frac{1}{n} \sum_{i=3}^{n-k} e_1^2(t_i) - \{1 + \rho_e(k)\} \right| \geq \frac{\tau_{0;n}}{\gamma_e(0)} \right) \\ & + P \left( \left| \frac{1}{n} \sum_{i=3}^{n-k} e_2^2(t_i) - \{1 - \rho_e(k)\} \right| \geq \frac{\tau_{0;n}}{\gamma_e(0)} \right) \equiv \text{IV} + \text{V}, \end{aligned}$$

where  $e_1(t_i) = \{e_0(t_i) + e_0(t_{i+k})\}/\sqrt{2\gamma_e(0)}$  and  $e_2(t_i) = \{e_0(t_i) - e_0(t_{i+k})\}/\sqrt{2\gamma_e(0)}$ . Then,  $\{e_1(t_i)\}$  and  $\{e_2(t_i)\}$  are  $(2g_n + 2)$ -dependent time series with mean zero.

We divide  $\{e_1(t_i)\}_{i=3}^{n-k}$  into consecutive blocks with length  $2g_n + 2$ , i.e.  $\{e_1(t_3), \dots, e_1(t_{2g_n+4})\}$ ,  $\{e_1(t_{2g_n+5}), \dots, e_1(t_{4g_n+6})\}, \dots$ . So, there are  $q_n = \lceil (n-k-2)/(2g_n+2) \rceil$  blocks, where  $\lceil \cdot \rceil$  denotes the ceiling function. The length of the last block is less than  $2g_n + 2$ , if  $2g_n + 2$  is not a divisor of  $n - k - 2$ . Denote the sum of  $\{e_1^2(t_i) - \{1 + \rho_e(k)\}\}$  within these blocks by  $b_1, \dots, b_{q_n}$ . For example,  $b_1 = \sum_{i=3}^{2g_n+4} [e_1^2(t_i) - \{1 + \rho_e(k)\}]$ . Then,  $E(b_j) = 0$  for  $j = 1, \dots, q_n$ .  $\{b_1, b_3, b_5, \dots\}$  are independent, so are  $\{b_2, b_4, b_6, \dots\}$ . By Cauchy-Schwarz inequality and Condition B1, we have

$$E(b_1^2) \leq (2g_n + 2) \sum_{i=3}^{2g_n+4} E[\{e_1^2(t_i) - (1 + \rho_e(k))\}^2] \leq C(2g_n + 2)^2.$$

Similarly, we can show that

$$E(b_j^2) \leq C(2g_n + 2)^2 \quad \text{for any } j = 1, \dots, q_n.$$



Since  $\tau_{0;n}^{-1} = o(n/g_n^2)$ , there exists a constant  $L_1 > 0$ , such that for any  $n > L_1$  and  $k \in \{0, \dots, g_n\}$ ,

$$(k+2)\{1 + \rho_e(k)\} \leq (g_n + 2)\{1 + \rho_e(k)\} \leq 2(g_n + 2) \leq \frac{n\tau_{0;n}}{2\gamma_e(0)}.$$

Then, due to Markov inequality, for  $n > L_1$ , we have

$$\begin{aligned} \text{IV} &\leq \mathbb{P}\left(\left|\sum_{i=3}^{n-k} [e_1^2(t_i) - \{1 + \rho_e(k)\}]\right| + (k+2)\{1 + \rho_e(k)\} \geq \frac{n\tau_{0;n}}{\gamma_e(0)}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{i=3}^{n-k} [e_1^2(t_i) - \{1 + \rho_e(k)\}]\right| \geq \frac{n\tau_{0;n}}{2\gamma_e(0)}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{j=1,3,5,\dots} b_j\right| + \left|\sum_{j=2,4,6,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{2\gamma_e(0)}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{j=1,3,5,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{4\gamma_e(0)}\right) + \mathbb{P}\left(\left|\sum_{j=2,4,6,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{4\gamma_e(0)}\right) \\ &\leq \left\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\right\}^2 E\left\{\left(\sum_{j=1,3,5,\dots} b_j\right)^2\right\} + \left\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\right\}^2 E\left\{\left(\sum_{j=2,4,6,\dots} b_j\right)^2\right\} \\ &= \left\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\right\}^2 \left\{\sum_{j=1}^{q_n} E(b_j^2)\right\} = O(g_n^3/(n\tau_{0;n}^2)). \end{aligned}$$

Similarly we can show,  $V = O(g_n^3/(n\tau_{0;n}^2))$ . Hence, we finish the proof. ■

**Lemma 3** Under Condition A1, for any positive sequence  $\tau_{0;n}$  and  $k \in \{0, 1, \dots, g_n\}$ ,

$$\mathbb{P}\left(\frac{1}{n} \left|\sum_{i=3}^{n-k} \{\tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) - \tilde{\gamma}_e(k)\}\right| \geq \tau_{0;n}\right) \leq C \frac{g_n^3}{n\tau_{0;n}^2},$$

where  $\tilde{e}_0(t_i) = \sum_{j=-g_n+2}^{g_n+2} \psi_j w_{i-j}$ ,  $\psi_j \equiv \psi_{n;j} = \phi_{n;j} - 2\phi_{n;j-1} + \phi_{n;j-2}$  and  $\tilde{\gamma}_e(k) = \text{cov}\{\tilde{e}_0(t_i), \tilde{e}_0(t_{i+k})\}$ .

*Proof:* Since  $\tilde{e}_0(t_i) = \sum_{j=-g_n+2}^{g_n+2} \psi_j w_{i-j}$ ,  $\{\tilde{e}_0(t_i)\}$  is  $(2g_n + 4)$ -dependent. For any  $k = 0, 1, \dots, g_n$ ,

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{n} \left|\sum_{i=3}^{n-k} \{\tilde{e}_0(t_i)\tilde{e}_0(t_{i+k}) - \tilde{\gamma}_e(k)\}\right| \geq \tau_{0;n}\right) \\ &= \mathbb{P}\left(\left|\sum_{i=3}^{n-k} [\{\tilde{e}_0(t_i) + \tilde{e}_0(t_{i+k})\}^2 - \{2\tilde{\gamma}_e(0) + 2\tilde{\gamma}_e(k)\}]\right. \right. \\ &\quad \left. \left. - \sum_{i=3}^{n-k} [\{\tilde{e}_0(t_i) - \tilde{e}_0(t_{i+k})\}^2 - \{2\tilde{\gamma}_e(0) - 2\tilde{\gamma}_e(k)\}]\right| \geq 4n\tau_{0;n}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{i=3}^{n-k} [\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}]\right| \geq \frac{n\tau_{0;n}}{\tilde{\gamma}_e(0)}\right) \end{aligned}$$

$$+\mathbb{P}\left(\left|\sum_{i=3}^{n-k}[\tilde{e}_2^2(t_i) - \{1 - \tilde{\rho}_e(k)\}]\right| \geq \frac{n\tau_{0;n}}{\tilde{\gamma}_e(0)}\right), \quad (\text{S.18})$$

where  $\tilde{\rho}_e(k) = \tilde{\gamma}_e(k)/\tilde{\gamma}_e(0)$ ,  $\tilde{e}_1(t_i) = \{\tilde{e}_0(t_i) + \tilde{e}_0(t_{i+k})\}/\{2\tilde{\gamma}_e(0)\}^{1/2}$  and  $\tilde{e}_2(t_i) = \{\tilde{e}_0(t_i) - \tilde{e}_0(t_{i+k})\}/\{2\tilde{\gamma}_e(0)\}^{1/2}$ . Then,  $\{\tilde{e}_1(t_i)\}$  and  $\{\tilde{e}_2(t_i)\}$  are  $(3g_n + 4)$ -dependent with mean zero.

Divide  $\{\tilde{e}_1(t_i)\}_{i=3}^{n-k}$  into non-overlapped consecutive blocks with length  $3g_n + 4$ , i.e.,  $\{\tilde{e}_1(t_3), \dots, \tilde{e}_1(t_{3g_n+6})\}$ ,  $\{\tilde{e}_1(t_{3g_n+7}), \dots, \tilde{e}_1(t_{6g_n+10})\}, \dots$ . There are  $q_n = \lceil (n - k - 2)/(3g_n + 4) \rceil$  blocks, where  $\lceil \cdot \rceil$  denotes the ceiling function. The length of the last block is less than  $3g_n + 4$ , if  $3g_n + 4$  is not a divisor of  $n - k - 2$ . Denote by  $b_j$  the sum of  $[\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}]$  within the  $j$ th block, for  $j = 1, \dots, q_n$ . For example,  $b_1 = \sum_{i=3}^{3g_n+6} [\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}]$ . Then,  $E(b_j) = 0$  for  $j = 1, \dots, q_n$ . We can show that  $\{b_1, b_3, b_5, \dots\}$  are independent, and so are  $\{b_2, b_4, b_6, \dots\}$ .

Since  $\tilde{\gamma}_e(0) = \sigma_w^2 \sum_{j=-g_n-2}^{g_n+2} \psi_j^2 = O(g_n)$  and

$$E\{\tilde{e}_0^4(t_i)\} = \sum_{j=-g_n-2}^{g_n+2} \psi_j^4 E(w_{i-j}^4) + \sum_{-g_n-2 \leq j \neq j' \leq g_n+2} \psi_j^2 \psi_{j'}^2 \sigma_w^4 = O\left(\left(\sum_{j=-g_n-2}^{g_n+2} \psi_j^2\right)^2\right),$$

we have

$$\begin{aligned} E[|\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}|^2] &= E[\{\tilde{e}_0(t_i) + \tilde{e}_0(t_{i+k})\}^4 / \{4\tilde{\gamma}_e^2(0)\} - \{1 + \tilde{\rho}_e(k)\}^2] \\ &= O\left(\left(\sum_{j=-g_n-2}^{g_n+2} \psi_j^2\right)^2 / \left\{4\sigma_w^4 \left(\sum_{j=-g_n-2}^{g_n+2} \psi_j^2\right)^2\right\}\right) - \{1 + \tilde{\rho}_e(k)\}^2 = O(1). \end{aligned}$$

By Cauchy-Schwarz inequality,  $E(b_1^2) \leq (3g_n + 4) \sum_{i=3}^{3g_n+6} E[|\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}|^2] = O(g_n^2)$ . Similarly, we can show that  $E(b_j^2) = O(g_n^2)$  for  $j = 1, \dots, q_n$ , which together with Markov inequality implies

$$\begin{aligned} &\mathbb{P}\left(\left|\sum_{i=3}^{n-k}[\tilde{e}_1^2(t_i) - \{1 + \tilde{\rho}_e(k)\}]\right| \geq \frac{n\tau_{0;n}}{\tilde{\gamma}_e(0)}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{j=1,3,5,\dots} b_j\right| + \left|\sum_{j=2,4,6,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{\tilde{\gamma}_e(0)}\right) \\ &\leq \mathbb{P}\left(\left|\sum_{j=1,3,5,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{2\tilde{\gamma}_e(0)}\right) + \mathbb{P}\left(\left|\sum_{j=2,4,6,\dots} b_j\right| \geq \frac{n\tau_{0;n}}{2\tilde{\gamma}_e(0)}\right) \\ &\leq \left\{\frac{2\tilde{\gamma}_e(0)}{n\tau_{0;n}}\right\}^2 E\left\{\left(\sum_{j=1,3,5,\dots} b_j\right)^2\right\} + \left\{\frac{2\tilde{\gamma}_e(0)}{n\tau_{0;n}}\right\}^2 E\left\{\left(\sum_{j=2,4,6,\dots} b_j\right)^2\right\} \\ &= \left\{\frac{2\tilde{\gamma}_e(0)}{n\tau_{0;n}}\right\}^2 \left\{\sum_{j=1}^{q_n} E(b_j^2)\right\} = O\left(\frac{g_n^2}{n^2\tau_{0;n}^2} q_n g_n^2\right) = O\left(\frac{g_n^3}{n\tau_{0;n}^2}\right). \quad (\text{S.19}) \end{aligned}$$

Similar arguments lead to

$$\mathbb{P}\left(\left|\sum_{i=3}^{n-k}[\tilde{e}_2^2(t_i) - \{1 - \tilde{\rho}_e(k)\}]\right| \geq \frac{n\tau_{0;n}}{\tilde{\gamma}_e(0)}\right) = O\left(\frac{g_n^3}{n\tau_{0;n}^2}\right). \quad (\text{S.20})$$

By (S.18), (S.19) and (S.20), we complete the proof. ■

**Lemma 4** *Under Conditions A2, A4 and A5, for any constant  $\tau_1 > 0$  and  $1 \leq i, j \leq \ell$ ,*

$$\mathbb{P}(\|\widehat{\Sigma}_{i,j} - \Sigma_{i,j}\|_\infty > \tau_1) \leq C/n,$$

where  $\widehat{\Sigma}_{i,j} = (\mathbf{D}_1 \mathbf{S}_i)^T (\mathbf{D}_1 \mathbf{S}_j) / n$  and  $\Sigma_{i,j}$  is defined in Condition A2.

*Proof:* For notational simplicity, define  $s_{r;i} = s_i(r/n)$ , for  $r = 0, \dots, n-1$ .

Let  $\mathbf{J}$  be an  $m \times m$  matrix defined as

$$\mathbf{J}(u, v) = \begin{cases} n^{-1} \sum_{r=0}^{n-v+u-2} (s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i}), & \text{if } 1 \leq u < v \leq m, \\ n^{-1} \sum_{r=0}^{n-u+v-2} (s_{r+1;i} - s_{r;i})(s_{r+u-v+1;j} - s_{r+u-v;j}), & \text{if } 1 \leq v \leq u \leq m. \end{cases}$$

For  $1 \leq u < v \leq m$ ,

$$\begin{aligned} & |\widehat{\Sigma}_{i,j}(u, v) - \mathbf{J}(u, v)| \\ &= \left| \frac{1}{n} \sum_{r=-1}^{n-v-1} (s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i}) \right. \\ &\quad \left. - \frac{1}{n} \sum_{r=0}^{n-v+u-2} (s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i}) \right| \\ &\leq \frac{1}{n} \left\{ |s_{0;j}(s_{v-u;i} - s_{v-u-1;i})| + \sum_{r=n-v-1}^{n-v+u-2} |(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i})| \right\} \\ &\equiv \mathbf{K}(u, v), \end{aligned} \tag{S.21}$$

where  $s_{-1;i} = s_{-1;j} = 0$  and  $\mathbf{K}$  is an  $m \times m$  matrix. Similarly, for  $1 \leq v \leq u \leq m$ ,

$$\begin{aligned} & |\widehat{\Sigma}_{i,j}(u, v) - \mathbf{J}(u, v)| \\ &\leq \frac{1}{n} \left\{ |s_{0;i}(s_{u-v;j} - s_{u-v-1;j})| + \sum_{r=n-u-1}^{n-u+v-2} |(s_{r+1;i} - s_{r;i})(s_{r+u-v+1;j} - s_{r+u-v;j})| \right\} \\ &\equiv \mathbf{K}(u, v). \end{aligned} \tag{S.22}$$

By Markov inequality and Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{P}\left(\sum_{1 \leq u < v \leq m} \mathbf{K}(u, v) > \frac{\tau_1}{4}\right) &\leq \frac{16}{\tau_1^2 n^2} E\left(\left[\sum_{1 \leq u < v \leq m} \left\{ |s_{0;j}(s_{v-u;i} - s_{v-u-1;i})| \right. \right. \right. \\ &\quad \left. \left. \left. + \sum_{r=n-v-1}^{n-v+u-2} |(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i})| \right\} \right]^2\right) \\ &= O(1/n^2), \end{aligned}$$

and similarly, we can show  $\mathbb{P}\left(\sum_{1 \leq v \leq u \leq m} \mathbf{K}(u, v) > \tau_1/4\right) = O(1/n^2)$ .

Since, by Condition A4,  $\Sigma_{i,j}$  is a Toeplitz matrix and so is  $\mathbf{J}$ , we have  $\|\mathbf{J} - \Sigma_{i,j}\|_\infty \leq \|\mathbf{z}_{1,m}^T(\mathbf{J} - \Sigma_{i,j})\|_1 + \|\mathbf{z}_{m,m}^T(\mathbf{J} - \Sigma_{i,j})\|_1 = \sum_{v=1}^m |\mathbf{J}(1,v) - \Sigma_{i,j}(1,v)| + \sum_{v=1}^m |\mathbf{J}(m,v) - \Sigma_{i,j}(m,v)|$ . From (S.21) and (S.22),

$$\begin{aligned}
& \mathbb{P}(\|\widehat{\Sigma}_{i,j} - \Sigma_{i,j}\|_\infty > \tau_1) \leq \mathbb{P}(\|\mathbf{J} - \Sigma_{i,j}\|_\infty + \|\widehat{\Sigma}_{i,j} - \mathbf{J}\|_\infty > \tau_1) \\
& \leq \mathbb{P}(\|\mathbf{J} - \Sigma_{i,j}\|_\infty > \tau_1/2) + \mathbb{P}(\|\widehat{\Sigma}_{i,j} - \mathbf{J}\|_\infty > \tau_1/2) \\
& \leq \mathbb{P}\left(\sum_{v=1}^m |\mathbf{J}(1,v) - \Sigma_{i,j}(1,v)| + \sum_{v=1}^m |\mathbf{J}(m,v) - \Sigma_{i,j}(m,v)| > \frac{\tau_1}{2}\right) \\
& \quad + \mathbb{P}\left(\sum_{1 \leq u, v \leq m} \mathbf{K}(u,v) > \frac{\tau_1}{2}\right) \\
& \leq \sum_{v=1}^m \mathbb{P}(|\mathbf{J}(1,v) - \Sigma_{i,j}(1,v)| > \tau_1/(4m)) \\
& \quad + \sum_{v=1}^m \mathbb{P}(|\mathbf{J}(m,v) - \Sigma_{i,j}(m,v)| > \tau_1/(4m)) + O(1/n^2). \tag{S.23}
\end{aligned}$$

Following basically the same method in the proof of Lemma 3, we can show  $\mathbb{P}(|\mathbf{J}(1,v) - \Sigma_{i,j}(1,v)| > \tau_1/(4m)) = O(1/n)$  and  $\mathbb{P}(|\mathbf{J}(m,v) - \Sigma_{i,j}(m,v)| > \tau_1/(4m)) = O(1/n)$ . By (S.23) we complete the proof. ■

**Lemma 5** Under Conditions A2, A4 and A5, for any constant  $\tau_1 > 0$ ,

$$\mathbb{P}(\|\widehat{\Sigma}_1 - \Sigma_1\| > \tau_1) \leq C/n,$$

where  $\widehat{\Sigma}_1 = (\mathbf{D}_1 \mathbf{S})^T (\mathbf{D}_1 \mathbf{S})/n$  and  $\Sigma_1$  is defined in Condition A2.

*Proof:*  $\widehat{\Sigma}_1$  could be expressed as a block matrix, i.e.,  $\widehat{\Sigma}_1 = (\widehat{\Sigma}_{i,j})_{i,j=1}^\ell$  where  $\widehat{\Sigma}_{i,j} = (\mathbf{D}_1 \mathbf{S}_i)^T (\mathbf{D}_1 \mathbf{S}_j)/n$ . Since  $\widehat{\Sigma}_1$  and  $\Sigma_1$  are symmetric matrices,

$$\begin{aligned}
& \mathbb{P}(\|\widehat{\Sigma}_1 - \Sigma_1\| > \tau_1) \leq \mathbb{P}(\|\widehat{\Sigma}_1 - \Sigma_1\|_\infty > \tau_1) \leq \mathbb{P}\left(\sum_{1 \leq i, j \leq \ell} \|\widehat{\Sigma}_{i,j} - \Sigma_{i,j}\|_\infty > \tau_1\right) \\
& \leq \sum_{1 \leq i, j \leq \ell} \mathbb{P}(\|\widehat{\Sigma}_{i,j} - \Sigma_{i,j}\|_\infty > \tau_1/\ell^2) = O(1/n).
\end{aligned}$$

The last inequality is derived from Lemma 4. ■

**Lemma 6** Under Conditions A1 (or B1) and A2–A5, for any  $\tau_2 \equiv \tau_{2;n} > 0$ ,

$$\mathbb{P}(\|(\mathbf{D}_1 \mathbf{S})^T \boldsymbol{\epsilon}_1\|^2 \geq \tau_2) \leq Cng_n^2/\tau_2,$$

where  $\boldsymbol{\epsilon}_1 = \mathbf{D}_1 \mathbf{d} + \mathbf{D}_1 \boldsymbol{\epsilon}$  as in Notation 1.

*Proof:* First, we will show the result under Conditions A1–A5. Let  $(\eta_1, \dots, \eta_{\ell m})^T = (\mathbf{D}_1 \mathbf{S})^T \boldsymbol{\epsilon}_1$  and  $\vartheta_{i,j} = \mathbf{z}_{i,n-1}^T (\mathbf{D}_1 \mathbf{S}) \mathbf{z}_{j,\ell m}$ . For  $j = 1, \dots, \ell m$ ,  $\eta_j = \sum_{i=1}^{n-1} \epsilon_1(t_{i+1}) \vartheta_{i,j} = \sum_{i=1}^{n-1} \{\epsilon(t_{i+1}) - \epsilon(t_i)\} \vartheta_{i,j} + \sum_{i=1}^{n-1} \{d(t_{i+1}) - d(t_i)\} \vartheta_{i,j}$ .

$$\mathbb{P}(\|(\mathbf{D}_1 \mathbf{S})^T \boldsymbol{\epsilon}_1\|^2 \geq \tau_2) = \mathbb{P}\left(\sum_{j=1}^{\ell m} \eta_j^2 \geq \tau_2\right) \leq \sum_{j=1}^{\ell m} \mathbb{P}(\eta_j^2 \geq \tau_2/(\ell m))$$

$$\begin{aligned}
&\leq \sum_{j=1}^{\ell m} \mathbb{P} \left( \left| \sum_{i=1}^{n-1} \{\epsilon(t_{i+1}) - \epsilon(t_i)\} \vartheta_{i,j} \right| \geq \frac{\tau_2^{1/2}}{2(\ell m)^{1/2}} \right) \\
&\quad + \sum_{j=1}^{\ell m} \mathbb{P} \left( \left| \sum_{i=1}^{n-1} \{d(t_{i+1}) - d(t_i)\} \vartheta_{i,j} \right| \geq \frac{\tau_2^{1/2}}{2(\ell m)^{1/2}} \right) \\
&\equiv \text{I} + \text{II}.
\end{aligned} \tag{S.24}$$

For term I, define  $\phi_k^* = \phi_{n;k} - \phi_{n;k-1}$ , so  $\epsilon(t_{i+1}) - \epsilon(t_i) = \sum_{k=-\infty}^{\infty} \phi_k^* w_{i+1-k}$ . Then,

$$\sum_{i=1}^{n-1} \{\epsilon(t_{i+1}) - \epsilon(t_i)\} \vartheta_{i,j} = \sum_{i=1}^{n-1} \sum_{k=-\infty}^{\infty} \phi_k^* w_{i+1-k} \vartheta_{i,j} = \sum_{k=-\infty}^{\infty} \sum_{i=1}^{n-1} \phi_k^* w_{i+1-k} \vartheta_{i,j}. \tag{S.25}$$

For each  $k$  and  $j$ , define  $\beta_{k,j} = (\phi_k^* w_{2-k} \vartheta_{1,j}, \dots, \phi_k^* w_{n-k} \vartheta_{n-1,j})^T$ . Divide  $\beta_{k,j}$  into blocks with length  $g_s + 1$ , and hence there are  $q_n = \lceil (n-1)/(g_s + 1) \rceil$  blocks. The sum of the elements of  $\beta_{k,j}$  within the  $u$ th block is denoted by  $\kappa_{k,j,u}$ , for  $u = 1, \dots, q_n$ . For example,  $\kappa_{k,j,1} = \sum_{i=1}^{g_s+1} \phi_k^* w_{i+1-k} \vartheta_{i,j}$ . Then,  $E(\kappa_{k,j,u}) = 0$ ,  $E(\kappa_{k,j,u}^2) \leq (g_s + 1) \sum_{i=1}^{g_s+1} E\{\phi_k^* w_{i+1-k} \vartheta_{i,j}\}^2 = O((g_s + 1)^2 \phi_k^{*2})$ ,  $\{\kappa_{k,j,1}, \kappa_{k,j,3}, \dots\}$  are independent and so are  $\{\kappa_{k,j,2}, \kappa_{k,j,4}, \dots\}$ . Then,

$$E \left\{ \left( \sum_{i=1}^{n-1} \phi_k^* w_{i+1-k} \vartheta_{i,j} \right)^2 \right\} \leq 2E \left\{ \left( \sum_{u=1,3,5,\dots} \kappa_{k,j,u} \right)^2 \right\} + 2E \left\{ \left( \sum_{u=2,4,6,\dots} \kappa_{k,j,u} \right)^2 \right\} = O(n \phi_k^{*2}),$$

which together with (S.25) and Minkowski inequality implies

$$\begin{aligned}
&E \left[ \left| \sum_{i=1}^{n-1} \{\epsilon(t_{i+1}) - \epsilon(t_i)\} \vartheta_{i,j} \right|^2 \right] = E \left[ \left| \sum_{k=-\infty}^{\infty} \sum_{i=1}^{n-1} (\phi_k^* w_{i+1-k} \vartheta_{i,j}) \right|^2 \right] \\
&\leq \left[ \sum_{k=-\infty}^{\infty} \left\{ E \left( \left| \sum_{i=1}^{n-1} (\phi_k^* w_{i+1-k} \vartheta_{i,j}) \right|^2 \right) \right\}^{1/2} \right]^2 = O(n g_n^2).
\end{aligned} \tag{S.26}$$

By Markov inequality and (S.26),

$$\text{I} \leq \frac{4\ell m}{\tau_2} \sum_{j=1}^{\ell m} E \left[ \left| \sum_{i=1}^{n-1} \{\epsilon(t_{i+1}) - \epsilon(t_i)\} \vartheta_{i,j} \right|^2 \right] = O(n g_n^2 / \tau_2).$$

Similar arguments can be applied to show that  $\text{II} = O(n g_n^2 / \tau_2)$ . From (S.24), we complete the proof.

Next, we will provide the proof under Conditions B1 and A2–A5. Since  $\{\epsilon(t_{i+1}) - \epsilon(t_i)\}$  is  $(g_n + 1)$ -dependent and any column of  $\mathbf{D}_1 \mathbf{S}$  is  $(g_s + 1)$ -dependent, the vector  $\alpha_j = (\{\epsilon(t_2) - \epsilon(t_1)\} \vartheta_{1,j}, \dots, \{\epsilon(t_n) - \epsilon(t_{n-1})\} \vartheta_{n-1,j})^T$  is  $(g_n + g_s + 1)$ -dependent,  $j = 1, \dots, m$ . We divide  $\alpha_j$  into blocks with length  $g_n + g_s + 1$  as we did before. So, there are  $\lceil (n-1)/(g_n + g_s + 1) \rceil$  blocks. The sum of elements in  $\alpha_j$  within the  $w$ th block is denoted by  $f_{j,w}$ , for  $w = 1, \dots, \lceil (n-1)/(g_n + g_s + 1) \rceil$ . By Conditions B1 and A4, it is

easy to see that  $E(f_{j,w}^2) \leq Cg_n^3$ . Hence,

$$\begin{aligned} & \mathbb{P}\left(\left|\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j}\right|\geq\frac{\tau_2^{1/2}}{2(\ell m)^{1/2}}\right) \\ & \leq \mathbb{P}\left(\left|\sum_{w=1,3,5,\dots}f_{j,w}\right|\geq\frac{\tau_2^{1/2}}{4(\ell m)^{1/2}}\right)+\mathbb{P}\left(\left|\sum_{w=2,4,6,\dots}f_{j,w}\right|\geq\frac{\tau_2^{1/2}}{4(\ell m)^{1/2}}\right) \\ & \leq \frac{16\ell m}{\tau_2}\left\{E\left(\left|\sum_{w=1,3,5,\dots}f_{j,w}\right|^2\right)+E\left(\left|\sum_{w=2,4,6,\dots}f_{j,w}\right|^2\right)\right\}=O(n g_n^2/\tau_2) \end{aligned}$$

Since (S.24) still holds under Condition B1, similar arguments can be applied to show that  $\text{I} = O(n g_n^2/\tau_2)$  and  $\text{II} = O(n g_n^2/\tau_2)$ . From (S.24), we complete the proof.  $\blacksquare$

**Lemma 7** *Under Conditions A1 (or B1) and A2–A5, for any  $\tau_3 \equiv \tau_{3;n} > 0$ ,*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=3}^n\delta^2(t_i)\geq\tau_3\right)\leq\frac{Cg_n^2}{\tau_3 n}+\frac{C}{n},$$

where  $\delta(t_i)$  is the  $(i-2)$ th element of  $\boldsymbol{\delta} = \mathbf{D}_2\mathbf{S}(\mathbf{h} - \widehat{\mathbf{h}}_{\text{DBE}})$  as defined in Notation 2.

*Proof:* The proof is the same under either Condition A1 or B1.

Since  $\boldsymbol{\epsilon}_1 = \mathbf{D}_1\mathbf{d} + \mathbf{D}_1\boldsymbol{\epsilon}$ , from model (2.1),

$$\widehat{\mathbf{h}}_{\text{DBE}} = \{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{y}) = \mathbf{h} + \{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1.$$

Thus,

$$\begin{aligned} \sum_{i=3}^n\delta^2(t_i) &= \|\mathbf{D}_2\mathbf{S}(\mathbf{h} - \widehat{\mathbf{h}}_{\text{DBE}})\|^2 = \|\mathbf{D}_0(\mathbf{D}_1\mathbf{S})\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \\ &\leq \|\mathbf{D}_0\|^2\|(\mathbf{D}_1\mathbf{S})\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}\|^2\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \\ &\leq 4\|\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}\|\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 = 4n^{-1}\|\widehat{\boldsymbol{\Sigma}}_1^{-1}\|\times\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2, \end{aligned} \quad (\text{S.27})$$

where  $\widehat{\boldsymbol{\Sigma}}_1 = (\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})/n$  and  $\mathbf{D}_0$  is an  $(n-2) \times (n-1)$  matrix defined as

$$\mathbf{D}_0 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-2) \times (n-1)},$$

such that  $\mathbf{D}_2 = \mathbf{D}_0\mathbf{D}_1$ . By Condition A2, when  $2^{-1}\lambda_{\min}(\boldsymbol{\Sigma}_1) \geq \|\widehat{\boldsymbol{\Sigma}}_1 - \boldsymbol{\Sigma}_1\|$ ,

$$\frac{C}{2} \leq \frac{1}{2}\lambda_{\min}(\boldsymbol{\Sigma}_1) \leq \lambda_{\min}(\boldsymbol{\Sigma}_1) - \|\widehat{\boldsymbol{\Sigma}}_1 - \boldsymbol{\Sigma}_1\| \leq \lambda_{\min}(\boldsymbol{\Sigma}_1) + \lambda_{\min}(\widehat{\boldsymbol{\Sigma}}_1 - \boldsymbol{\Sigma}_1) \leq \lambda_{\min}(\widehat{\boldsymbol{\Sigma}}_1),$$

and thus,  $\|\widehat{\Sigma}_1^{-1}\| = 1/\lambda_{\min}(\widehat{\Sigma}_1) \leq 2/C$ , which together with (S.27) implies

$$\begin{aligned}
& \mathbb{P}\left(\frac{1}{n} \sum_{i=3}^n \delta^2(t_i) \geq \tau_3\right) \leq \mathbb{P}(\|\widehat{\Sigma}_1^{-1}\| \|\mathbf{D}_1 \mathbf{S}\|^T \boldsymbol{\epsilon}_1\|^2 \geq \tau_3 n^2/4) \\
&= \mathbb{P}(\|\widehat{\Sigma}_1^{-1}\| \|\mathbf{D}_1 \mathbf{S}\|^T \boldsymbol{\epsilon}_1\|^2 \geq \tau_3 n^2/4, \|\widehat{\Sigma}_1 - \Sigma_1\| \leq \lambda_{\min}(\Sigma_1)/2) \\
&\quad + \mathbb{P}(\|\widehat{\Sigma}_1^{-1}\| \|\mathbf{D}_1 \mathbf{S}\|^T \boldsymbol{\epsilon}_1\|^2 \geq \tau_3 n^2/4, \|\widehat{\Sigma}_1 - \Sigma_1\| > \lambda_{\min}(\Sigma_1)/2) \\
&\leq \mathbb{P}(\|\mathbf{D}_1 \mathbf{S}\|^T \boldsymbol{\epsilon}_1\|^2 \geq C\tau_3 n^2/8) + \mathbb{P}(\|\widehat{\Sigma}_1 - \Sigma_1\| > C/2) \\
&\equiv \text{I} + \text{II}.
\end{aligned} \tag{S.28}$$

By Lemma 6, if we take  $\tau_{2,n} = C\tau_3 n^2/8$ ,  $\text{I} = O(g_n^2/(\tau_3 n))$ . From Lemma 5, by choosing  $\tau_1 = C/2$ ,  $\text{II} = O(1/n)$ . From (S.28), we complete the proof. ■

**Lemma 8** *Under Conditions A1 (or B1) and A2–A5, for  $\varepsilon_0 \equiv \varepsilon_{0,n} > 0$  that satisfies  $\varepsilon_0 = O(g_n^6)$ ,  $\varepsilon_0 n/g_n^{11} \rightarrow \infty$  and  $\varepsilon_0^2 n^2/g_n^{11} \rightarrow \infty$ , and for any  $k \in \{0, \dots, g_n\}$ ,*

$$\mathbb{P}(|\widehat{\gamma}_e(k) - \gamma_e(k)| \geq \varepsilon_0/g_n^5) \leq Cg_n^{13}/(n\varepsilon_0^2).$$

*Proof:* The proof is the same under either Condition A1 or B1.

Since  $\widehat{e}(t_i) = e(t_i) + \delta(t_i)$ , we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=3}^{n-k} \widehat{e}(t_i) \widehat{e}(t_{i+k}) = \frac{1}{n} \sum_{i=3}^{n-k} \{e(t_i) + \delta(t_i)\} \{e(t_{i+k}) + \delta(t_{i+k})\} \\
&= \frac{1}{n} \sum_{i=3}^{n-k} e(t_i) e(t_{i+k}) + \frac{1}{n} \sum_{i=3}^{n-k} e(t_i) \delta(t_{i+k}) + \frac{1}{n} \sum_{i=3}^{n-k} e(t_{i+k}) \delta(t_i) + \frac{1}{n} \sum_{i=3}^{n-k} \delta(t_i) \delta(t_{i+k}).
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
& \mathbb{P}\left(|\widehat{\gamma}_e(k) - \gamma_e(k)| \geq \frac{\varepsilon_0}{g_n^5}\right) = \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{\widehat{e}(t_i) \widehat{e}(t_{i+k})\} - \gamma_e(k)\right| \geq \frac{\varepsilon_0}{g_n^5}\right) \\
&\leq \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e(t_i) e(t_{i+k})\} - \gamma_e(k)\right| + \frac{2}{n} \left\{ \sum_{i=3}^n e^2(t_i) \sum_{i=3}^n \delta^2(t_i) \right\}^{1/2} + \frac{1}{n} \sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0}{g_n^5}\right) \\
&\leq \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e(t_i) e(t_{i+k})\} - \gamma_e(k)\right| \geq \frac{\varepsilon_0}{3g_n^5}\right) + \mathbb{P}\left(\frac{2}{n} \left\{ \sum_{i=3}^n e^2(t_i) \sum_{i=3}^n \delta^2(t_i) \right\}^{1/2} \geq \frac{\varepsilon_0}{3g_n^5}\right) \\
&\quad + \mathbb{P}\left(\frac{1}{n} \sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0}{3g_n^5}\right) \\
&\equiv \text{I} + \text{II} + \text{III}.
\end{aligned} \tag{S.29}$$

From Conditions A3 and A5, we have  $|d_0(t_i)| \leq 2C/n^2$ , where  $d_0(t_i) = d(t_i) - 2d(t_{i-1}) + d(t_{i-2})$ . Since  $e(t_i) = e_0(t_i) + d_0(t_i)$ , by Cauchy-Schwarz inequality, for large  $n$ ,

$$\text{I} \leq \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=3}^{n-k} \{e_0(t_i) e_0(t_{i+k})\} - \gamma_e(k)\right| \geq \frac{\varepsilon_0}{4g_n^5}\right)$$

$$\begin{aligned}
& +\mathbb{P}\left(\frac{2}{n}\left\{\sum_{i=3}^n e_0^2(t_i)\sum_{i=3}^n d_0^2(t_i)\right\}^{1/2} + \frac{1}{n}\sum_{i=3}^n d_0^2(t_i) \geq \frac{\varepsilon_0}{12g_n^5}\right) \\
& \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\right| \geq \frac{\varepsilon_0}{4g_n^5}\right) \\
& \quad +\mathbb{P}\left(\frac{2}{n}\left\{\sum_{i=3}^n e_0^2(t_i)\sum_{i=3}^n d_0^2(t_i)\right\}^{1/2} \geq \frac{\varepsilon_0}{24g_n^5}\right) \\
& \equiv \mathbb{I}_1 + \mathbb{I}_2. \tag{S.30}
\end{aligned}$$

The last inequality in (S.30) is true, when  $n$  is large enough, such that  $n^4/g_n^5 > 96C^2/\varepsilon_0$ , which implies  $n^{-1}\sum_{i=3}^n d_0^2(t_i) < \varepsilon_0/(24g_n^5)$ .

For term  $\mathbb{I}_2$  in (S.30), when  $n$  is large enough,

$$\begin{aligned}
\mathbb{I}_2 & = \mathbb{P}\left(\frac{2}{n}\left\{\sum_{i=3}^n e_0^2(t_i)\sum_{i=3}^n d_0^2(t_i)\right\}^{1/2} \geq \frac{\varepsilon_0}{24g_n^5}\right) = \mathbb{P}\left(\sum_{i=3}^n e_0^2(t_i)\sum_{i=3}^n d_0^2(t_i) \geq \frac{\varepsilon_0^2 n^2}{2304g_n^{10}}\right) \\
& \leq \mathbb{P}\left(\sum_{i=3}^n e_0^2(t_i) \geq \frac{\varepsilon_0^2 n^5}{9216C^2g_n^{10}}\right) \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^n e_0^2(t_i) - \gamma_e(0)\right| \geq \frac{\varepsilon_0^2 n^4}{18432C^2g_n^{10}}\right) \\
& \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^n e_0^2(t_i) - \gamma_e(0)\right| \geq \frac{\varepsilon_0}{4g_n^5}\right). \tag{S.31}
\end{aligned}$$

The last two inequalities in (S.31) are true, when  $n$  is large enough for the following inequalities to hold,  $n^4/g_n^{10} > 18432C^2\gamma_e(0)/\varepsilon_0^2$  and  $n^4/g_n^5 > 4608C^2/\varepsilon_0$ , which imply  $\gamma_e(0) < \varepsilon_0^2 n^4/(18432C^2g_n^{10})$  and  $\varepsilon_0/(4g_n^5) < \varepsilon_0^2 n^4/(18432C^2g_n^{10})$  respectively.

From the assumptions that  $\varepsilon_0 n/g_n^{11} \rightarrow \infty$  and  $\varepsilon_0^2 n^2/g_n^{11} \rightarrow \infty$ , we can always choose a constant  $L_1$ , such that, for any  $n > L_1$ , the following inequalities hold:  $n^4/g_n^5 > 96C^2/\varepsilon_0$ ,  $n^4/g_n^{10} > 18432C^2\gamma_e(0)/\varepsilon_0^2$  and  $n^4/g_n^5 > 4608C^2/\varepsilon_0$ , which imply that (S.30) and (S.31) hold. Therefore, for  $n > L_1$ , from (S.30), (S.31), by choosing  $\tau_{0,n} = \varepsilon_0/(4g_n^5)$  in Lemma 2,

$$\begin{aligned}
\mathbb{I} & \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\right| \geq \frac{\varepsilon_0}{4g_n^5}\right) + \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^n e_0^2(t_i) - \gamma_e(0)\right| \geq \frac{\varepsilon_0}{4g_n^5}\right) \\
& = O(g_n^{13}/(n\varepsilon_0^2)). \tag{S.32}
\end{aligned}$$

For term  $\mathbb{II}$ ,

$$\begin{aligned}
\mathbb{II} & = \mathbb{P}\left(\frac{1}{n}\left\{\frac{1}{n}\sum_{i=3}^n e^2(t_i) - \gamma_e(0) + \gamma_e(0)\right\}\sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0^2}{36g_n^{10}}\right) \\
& \leq \mathbb{P}\left(\frac{1}{n}\left|\frac{1}{n}\sum_{i=3}^n e^2(t_i) - \gamma_e(0)\right|\sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0^2}{72g_n^{10}}\right) \\
& \quad +\mathbb{P}\left(\frac{1}{n}\gamma_e(0)\sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0^2}{72g_n^{10}}\right) \\
& \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=3}^n e^2(t_i) - \gamma_e(0)\right| \geq \frac{\varepsilon_0}{3g_n^5}\right) + \mathbb{P}\left(\frac{1}{n}\sum_{i=3}^n \delta^2(t_i) \geq \frac{\varepsilon_0}{24g_n^5}\right)
\end{aligned}$$



$$\begin{aligned}
& +\mathrm{P}\left(\frac{1}{n}\gamma_e(0)\sum_{i=3}^n\delta^2(t_i)\geq\frac{\varepsilon_0^2}{72g_n^{10}}\right) \\
\equiv & \quad \Pi_1 + \Pi_2 + \Pi_3.
\end{aligned} \tag{S.33}$$

From the assumption  $0 < \varepsilon_0 = O(g_n^6)$ , we can first choose  $\tau_{3;n} = \varepsilon_0^2 / \{72\gamma_e(0)g_n^{10}\}$  in Lemma 7 and get  $\Pi_3 = O(g_n^{13}/(n\varepsilon_0^2)) + O(1/n) = O(g_n^{13}/(n\varepsilon_0^2))$ . Then, from Lemma 7, by taking  $\tau_{3;n} = \varepsilon_0/(24g_n^5)$ ,  $\Pi_2 = O(g_n^7/(n\varepsilon_0)) + O(1/n) = O(g_n^7/(n\varepsilon_0))$ . Based on the proof for term I,  $\Pi_1 = O(g_n^{13}/(n\varepsilon_0^2))$ . By (S.33) and the assumption that  $\varepsilon_0 = O(g_n^6)$ ,

$$\Pi = O(g_n^{13}/(n\varepsilon_0^2)) + O(g_n^7/(n\varepsilon_0)) + O(g_n^{13}/(n\varepsilon_0^2)) = O(g_n^{13}/(n\varepsilon_0^2)).$$

It's easy to see that

$$\text{III} \leq \Pi_2 = O(g_n^7/(n\varepsilon_0)) = O(g_n^{13}/(n\varepsilon_0^2)). \tag{S.34}$$

From (S.29) and (S.32)–(S.34), we complete the proof for  $n > L_1$ . The result for  $n \leq L_1$  is straightforward. ■

**Lemma 9** *Assume Conditions A2–A5 and that there exists  $\varepsilon \equiv \varepsilon_n > 0$  such that  $\varepsilon = O(1)$ ,  $\varepsilon^{1/2}n/g_n^{11} \rightarrow \infty$ ,  $\varepsilon n^2/g_n^{11} \rightarrow \infty$ . Under either of the following two assumptions:*

- *Condition A1 holds and  $(g_n + 2)^{3-\alpha_n} = o(\varepsilon^{1/2})$ ,*
- *Condition B1 holds,*

we have

$$\mathrm{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \geq \varepsilon) \leq Cg_n^{14}/(\varepsilon n).$$

*Proof:* The proof is the same under either Condition A1 or B1.

Since  $\mathbf{R}$  and  $\widehat{\mathbf{R}}$  are Toeplitz matrices,

$$\begin{aligned}
& \mathrm{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \geq \varepsilon) \leq \mathrm{P}\left(\left\{2\sum_{k=1}^{g_n}|\widehat{\rho}(k) - \rho(k)|\right\}^2 \geq \varepsilon\right) \\
= & \quad \mathrm{P}\left(\sum_{k=1}^{g_n}|\widehat{\rho}(k) - \rho(k)| \geq \varepsilon^{1/2}/2\right) = \mathrm{P}\left(\left\|\frac{\widehat{\gamma}}{\widehat{\gamma}(0)} - \frac{\gamma}{\gamma(0)}\right\|_1 \geq \varepsilon^{1/2}/2\right) \\
\leq & \quad \mathrm{P}\left(\left\|\frac{\widehat{\gamma}}{\widehat{\gamma}(0)} - \frac{\gamma}{\gamma(0)}\right\|_1 \geq \varepsilon^{1/2}/2, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\right) + \mathrm{P}(|\widehat{\gamma}(0) - \gamma(0)| \geq \gamma(0)/2) \\
\equiv & \quad \text{I} + \text{II}.
\end{aligned} \tag{S.35}$$

Since

$$\frac{\widehat{\gamma}}{\widehat{\gamma}(0)} - \frac{\gamma}{\gamma(0)} = (\widehat{\gamma} - \gamma)\left(\frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)}\right) + \gamma\left(\frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)}\right) + (\widehat{\gamma} - \gamma)\frac{1}{\gamma(0)},$$

then,

$$\left\| \frac{\widehat{\gamma}}{\widehat{\gamma}(0)} - \frac{\gamma}{\gamma(0)} \right\|_1 \leq \|\widehat{\gamma} - \gamma\|_1 \left| \frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)} \right| + \|\gamma\|_1 \left| \frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)} \right| + \|\widehat{\gamma} - \gamma\|_1 \frac{1}{\gamma(0)},$$

which implies that

$$\begin{aligned} \text{I} &\leq \mathbb{P}\left(\|\widehat{\gamma} - \gamma\|_1 \left| \frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)} \right| \geq \varepsilon^{1/2}/6, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\right) \\ &\quad + \mathbb{P}\left(\|\gamma\|_1 \left| \frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)} \right| \geq \varepsilon^{1/2}/6, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\right) \\ &\quad + \mathbb{P}\left(\|\widehat{\gamma} - \gamma\|_1 \frac{1}{\gamma(0)} \geq \varepsilon^{1/2}/6, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\right) \\ &\equiv \text{III} + \text{IV} + \text{V}. \end{aligned} \tag{S.36}$$

Let  $\varepsilon_0 = \varepsilon^{1/2}\gamma(0)/96$ . By Lemma 1, Remark 1 and the assumption that  $(g_n + 2)^{3-\alpha_n} = o(\varepsilon^{1/2})$ , we have  $\|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n + 1)| + |\gamma(g_n + 2)|\} = O(g_n^2(g_n + 2)^{1-\alpha_n}) = O((g_n + 2)^{3-\alpha_n}) = o(\varepsilon_0)$ . There exists a constant  $L_1 > 0$ , such that for  $n > L_1$ ,  $\|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n + 1)| + |\gamma(g_n + 2)|\} \leq 2\varepsilon_0$ . For  $n > L_1$ , from Lemma 1,

$$\begin{aligned} \text{V} &\leq \mathbb{P}(\|\widehat{\gamma} - \gamma\|_1 \geq \varepsilon^{1/2}\gamma(0)/6) \\ &\leq \mathbb{P}(\|A_{g_n}^{-1}(\widehat{\gamma}_e - \gamma_e)\|_1 + \|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n + 1)| + |\gamma(g_n + 2)|\} \geq 4\varepsilon_0) \\ &\leq \mathbb{P}(\|A_{g_n}^{-1}\|_1\|\widehat{\gamma}_e - \gamma_e\|_1 \geq 2\varepsilon_0) \leq \mathbb{P}(\|\widehat{\gamma}_e - \gamma_e\|_1 \geq 2\varepsilon_0/(Cg_n^4)). \end{aligned} \tag{S.37}$$

Similarly, we can show that, for  $n > L_1$ ,

$$\text{III} \leq \mathbb{P}(\|\widehat{\gamma} - \gamma\|_1 \geq \varepsilon^{1/2}\gamma(0)/6).$$

By Lemma 1, Remark 1 and the assumption that  $(g_n + 2)^{3-\alpha_n} = o(\varepsilon^{1/2})$ , there exists a constant  $L_2 > 0$ , such that, for  $n > L_2$ ,

$$\|\mathbf{z}_{1,g_n+1}^T A_{g_n}^{-1} B_{g_n}\|_\infty \{|\gamma(g_n + 1)| + |\gamma(g_n + 2)|\} \leq \varepsilon^{1/2}\gamma(0)/\{24(g_n + 1)\}.$$

For  $n > L_2$ , from Lemma 1,

$$\begin{aligned} \text{IV} &\leq \mathbb{P}\left(\left| \frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)} \right| \geq \varepsilon^{1/2}/\{6\gamma(0)(g_n + 1)\}, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\right) \\ &\leq \mathbb{P}(|\widehat{\gamma}(0) - \gamma(0)| \geq \varepsilon^{1/2}\gamma(0)/\{12(g_n + 1)\}) \\ &\leq \mathbb{P}(|\mathbf{z}_{1,g_n+1}^T A_{g_n}^{-1}(\widehat{\gamma}_e - \gamma_e)| + \|\mathbf{z}_{1,g_n+1}^T A_{g_n}^{-1} B_{g_n}\|_\infty \{|\gamma(g_n + 1)| + |\gamma(g_n + 2)|\} \\ &\quad \geq \varepsilon^{1/2}\gamma(0)/\{12(g_n + 1)\}) \\ &\leq \mathbb{P}(\|\mathbf{z}_{1,g_n+1}^T A_{g_n}^{-1}\|_\infty \|\widehat{\gamma}_e - \gamma_e\|_1 \geq \varepsilon^{1/2}\gamma(0)/\{24(g_n + 1)\}) \\ &\leq \mathbb{P}(\|\widehat{\gamma}_e - \gamma_e\|_1 \geq 2\varepsilon_0/(Cg_n^4)). \end{aligned}$$

By (S.36) and Lemma 8, for  $n > \max\{L_1, L_2\}$ ,

$$\text{I} \leq 3 \sum_{k=0}^{g_n} \mathbb{P}(|\widehat{\gamma}_e(k) - \gamma_e(k)| \geq \varepsilon_0/(Cg_n^5)) = O(g_n^{14}/(\varepsilon_0^2 n)) = O(g_n^{14}/(\varepsilon n)).$$

From the assumption that  $\varepsilon = O(1)$ , we have  $C\varepsilon_0 \leq \gamma(0)/2$  for some positive constant  $C$ . Following basically the same procedure in (S.37), there exists a constant  $L_3 > 0$ , such that, for  $n > L_3$ ,

$$\text{II} \leq \text{P}(\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_1 \geq \gamma(0)/2) \leq \text{P}(\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_1 \geq C\varepsilon_0) = O(g_n^{14}/(\varepsilon n)).$$

Now by (S.35), we complete the proof for  $n > \max\{L_1, L_2, L_3\}$ . The result for  $n \leq \max\{L_1, L_2, L_3\}$  could be easily derived. ■

**Proof of Theorem 1.** It is easy to see that

$$\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \leq 2\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 + 2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2 \equiv \text{I} + \text{II}.$$

For term II, from Remark 1, under Condition A1,

$$\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty \leq 2 \sum_{k=g_n+1}^{n-1} |\gamma(k)|/|\gamma(0)| = O\left(\sum_{k=g_n+1}^{n-1} (2k - g_n)^{1-\alpha_n}\right) = O((g_n + 2)^{2-\alpha_n}).$$

Therefore,

$$\text{II} = O_P((g_n + 2)^{4-2\alpha_n}).$$

From  $(g_n + 2)^{8+2\alpha_n}/n \rightarrow \infty$  and  $g_n^{14}/n = o(1)$ ,  $(g_n + 2)^{4-2\alpha_n} = o(g_n^{14}/n)$ , and hence,  $\text{II} = o_P(g_n^{14}/n)$ . Under Condition B1,  $\text{II} = 0$ .

Take  $\varepsilon_n = C^* g_n^{14}/n$ , where  $C^* > 0$  is a constant. From Lemma 9, under either Condition A1 or B1,  $\text{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \geq C^* g_n^{14}/n) \leq C/C^*$ . Thus,

$$\text{I} = O_P(g_n^{14}/n).$$

We complete the proof. ■

**Proof of Proposition 1.** From the proof of Theorem 1,  $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \leq 2\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 + 2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2$ , and  $\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty = o(1)$ . From Lemma 9, for any constant  $\varepsilon > 0$  and  $n$  large enough such that  $2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2 < \varepsilon/2$ ,

$$\text{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \geq \varepsilon) \leq \text{P}(2\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \geq \varepsilon/2) \leq Cg_n^{14}/n = o(1), \quad (\text{S.38})$$

because  $\alpha_n g_n \rightarrow \infty$  implies that  $(g_n + 2)^{3-\alpha_n} = o(1)$ . So, we complete the proof. ■

**Proof of Proposition 2.** The proof is the same under either Condition A1 or B1. We can show that

$$\lambda_{\min}(\widehat{\mathbf{R}}) \geq \lambda_{\min}(\mathbf{R}) + \lambda_{\min}(\widehat{\mathbf{R}} - \mathbf{R}) \geq \lambda_{\min}(\mathbf{R}) - \|\widehat{\mathbf{R}} - \mathbf{R}\| \geq \lambda_{\min}(\mathbf{R}) - \|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty.$$

From Condition A6 and Proposition 1, we can show that with probability tending to one,  $\lambda_{\min}(\widehat{\mathbf{R}}) > 0$ . Thus, we complete the proof. ■

**Proof of Proposition 3.** The proof is the same under either Condition A1 or B1. The proof is similar for  $\widehat{\mathbf{R}}_Z$  and  $\widehat{\mathbf{R}}_*$ . In the following, we will only give the proof for  $\widehat{\mathbf{R}}_*$ . Since  $\widehat{\mathbf{R}}_* - \mathbf{R} = (\widehat{\mathbf{R}} - \mathbf{R})\mathbf{I}(\widehat{\mathbf{R}} \succ 0, \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega) + (\mathbf{I}_n - \mathbf{R})\{1 - \mathbf{I}(\widehat{\mathbf{R}} \succ 0, \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega)\}$ ,  $\|\widehat{\mathbf{R}}_* - \mathbf{R}\|_\infty \leq \|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty + \|\mathbf{I}_n - \mathbf{R}\|_\infty\{1 - \mathbf{I}(\widehat{\mathbf{R}} \succ 0, \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega)\}$ . From the result of Theorem 1, it suffices to show  $\lim_{n \rightarrow \infty} \mathbf{P}(\widehat{\mathbf{R}} \succ 0, \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega) = 1$ .

By Condition A6, we can verify that there is a constant  $M_1 > 0$ , such that  $\|\mathbf{R}^{-1}\|_\infty < M_1$ . Define the event  $Q = \{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty \leq \xi\}$ , for some  $0 < \xi < \min\{1/M_1, c, 1/C\}$ , where  $c$  and  $C$  are constants in Condition A6. From Theorem 1,  $\lim_{n \rightarrow \infty} \mathbf{P}(Q) = 1$ . Following the proof of Theorem 6 in Cai and Zhou (2012),  $\|\widehat{\mathbf{R}}^{-1}\|_\infty$  is bounded on  $Q$ . Hence,  $\lim_{n \rightarrow \infty} \mathbf{P}(\|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega) \geq \lim_{n \rightarrow \infty} \mathbf{P}(\|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega, Q) = 1$ . Together with  $\lim_{n \rightarrow \infty} \mathbf{P}(\widehat{\mathbf{R}} \succ 0) = 1$  from Proposition 2, we can conclude  $\lim_{n \rightarrow \infty} \mathbf{P}(\widehat{\mathbf{R}} \succ 0, \|\widehat{\mathbf{R}}^{-1}\|_\infty \leq Dn^\omega) = 1$ . ■

**Proof of Theorem 2.** The proof is the same under either Condition A1 or B1. By Condition A6, we can verify that there is a constant  $M_1 > 0$ , such that  $\|\mathbf{R}^{-1}\|_\infty < M_1$ . Define the event  $Q = \{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty \leq \xi\}$ , for some  $0 < \xi < \min\{1/M_1, c, 1/C\}$ . Following the proof of Theorem 6 in Cai and Zhou (2012), we can show that, for  $n$  large enough,  $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty \leq C_0\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty$  on  $Q$  and  $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty \leq C_0n^\omega$  on  $Q^c$ , where  $C_0 > 0$  is a constant. Thus, from similar arguments in (S.38), for  $n$  large enough,

$$\begin{aligned} E(\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2) &= E\{\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2 \mathbf{I}(Q)\} + E\{\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2 \mathbf{I}(Q^c)\} \\ &\leq C_0^2 E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q)\} + C_0^2 n^{2\omega} \mathbf{P}(Q^c) \\ &= C_0^2 E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q)\} + O(g_n^{14}/n^{1-2\omega}). \end{aligned} \quad (\text{S.39})$$

Since by (S.38), for any constant  $\varepsilon > 0$ ,  $\mathbf{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \geq \varepsilon) = O(g_n^{14}/n) \rightarrow 0$ , we have  $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \xrightarrow{\mathbf{P}} 0$ , which implies  $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q) \xrightarrow{\mathbf{P}} 0$ . Then,

$$\begin{aligned} &E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q) \mathbf{I}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q) \geq \varepsilon)\} \\ &\leq [E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^4 \mathbf{I}(Q)\} \mathbf{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \geq \varepsilon)]^{1/2} \\ &\leq \{\xi^4 \mathbf{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \geq \varepsilon)\}^{1/2} = O(g_n^7/n^{1/2}) \rightarrow 0. \end{aligned}$$

By asymptotically uniform integrability, we have  $E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \mathbf{I}(Q)\} \rightarrow 0$ , which together with (S.39) implies  $E(\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2) \rightarrow 0$ . ■

**Proof of Proposition 4.** The proof is the same under either Condition A1 or B1. Following the proof in Proposition 3,  $\|\mathbf{R}^{-1}\|_\infty$  is bounded, and  $\|\widehat{\mathbf{R}}^{-1}\|_\infty$  is bounded on  $Q$  (defined in Proposition 3). Since  $\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1} = \widehat{\mathbf{R}}^{-1}(\mathbf{R} - \widehat{\mathbf{R}})\mathbf{R}^{-1}$ ,  $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty \leq \|\widehat{\mathbf{R}}^{-1}\|_\infty \|\mathbf{R} - \widehat{\mathbf{R}}\|_\infty \|\mathbf{R}^{-1}\|_\infty$  on  $Q$ , and hence,  $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty \leq C\|\mathbf{R} - \widehat{\mathbf{R}}\|_\infty$ . From the result in Theorem 1 and  $\mathbf{P}(Q^c) = o(1)$ ,  $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$ .

From the result in Proposition 2 that  $\lim_{n \rightarrow \infty} \mathbf{P}(\widehat{\mathbf{R}} \succ 0) = 1$ , it is easy to prove that  $\|\widehat{\mathbf{R}}_Z^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$ .

From  $\lim_{n \rightarrow \infty} \mathbb{P}(\widehat{\mathbf{R}} \succ 0) = 1$  and that  $\|\widehat{\mathbf{R}}^{-1}\|_\infty$  is bounded on  $Q$ , it's easy to show  $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$ . ■