Review of posterior consistency & convergence rates

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Working group meeting presentation

Overview of the slides

- Priors on density space
- Notions of neighborhood and distances
- Consistent tests
- Weak and strong posterior consistency - main conditions & applications
- Notion of rates of posterior convergence
- Main conditions
- Examples

Basic notations

- NP Bayes - priors on infinite dimensional space (density, regression function, conditional density etc)
- Examples - Dirichlet process, Gaussian process, Levy process etc
- Today - posterior consistency & rates in density estimation
- $\mathcal{X}$ - complete separable metric space ($\Re$ for our discussion), $\mathcal{B}$ Borel σ-field on $\mathcal{X}$
- $\mathcal{F}$ space of densities on $(\mathcal{X}, \mathcal{B})$ w.r.t. some dominating measure
- $Y_1, \ldots, Y_n \stackrel{\text{i.i.d.}}{\sim} f \in \mathcal{F}$, $f \sim \Pi$

Main questions

- The posterior distribution is the random measure

$$\Pi(B \mid y^n) = \frac{\int_B \prod_{i=1}^n f(y_i) d\Pi(f)}{\int_{\mathcal{F}} \prod_{i=1}^n f(y_i) d\Pi(f)}$$

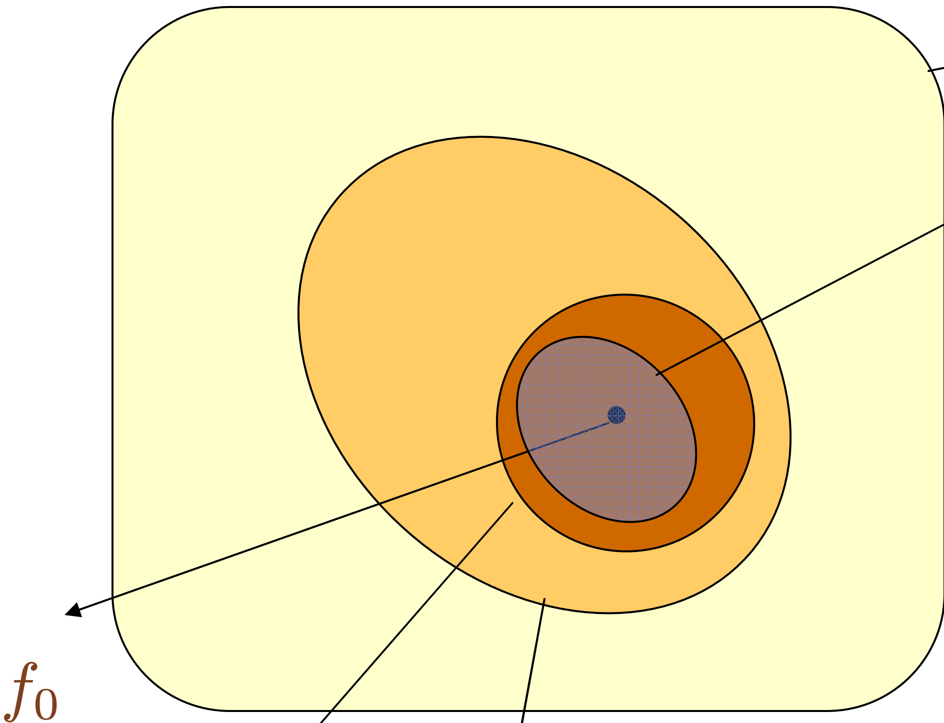
where B is a m'ble subset of $\mathcal{F}$ and $y^n = (y_1, \ldots, y_n)$

- Assume data sampled i.i.d. from $f_0 \in \mathcal{F}$
- Qn: does the posterior concentrate on arbitrary small neighborhoods of f_0 as $n \to \infty$? If so, at what rate? For which neighborhoods?
- First, need notions of distances and neighborhoods on density spaces

Distances & nbds on density space

- Weak convergence - $f_n \to f$ weakly if for any bounded continuous function ϕ, $\int \phi f_n \to \int \phi f$
- A weak nbd $W_\epsilon(f_0) = \{f \in \mathcal{F} : |\int \phi f - \int \phi f_0| < \epsilon\}$
- Strong or L_1 convergence - $f_n \to f$ in L_1 if $\int |f_n - f| \to 0$
- A strong nbd $S_\epsilon(f_0) = \{f \in \mathcal{F} : \int |f - f_0| = ||f - f_0||_1 < \epsilon\}$
- Also, $\mathrm{KL}(f_0, f) = \int f_0 \log(f_0/f)$, $h^2(f, f_0) = \int(\sqrt{f} - \sqrt{f_0})^2$
- A KL nbd $KL_\epsilon(f_0) = \{f \in \mathcal{F} : \mathrm{KL}(f_0, f) < \epsilon\}$
- Entropy of $\mathcal{F}_0 \subset \mathcal{F} := \log N(\epsilon, \mathcal{F}_0, ||\cdot||_1)$ is log min. number of balls of radius ϵ in the metric d required to cover $\mathcal{F}_0$.
- Interplay among these distances crucial, list of common inequalities in appendix

Weak / strong neighborhood / consistency



$$K_\epsilon(f_0) = \{f : f \in \mathcal{F}, KL(f_0, f) < \epsilon^2/4\}$$

Weak consistency:

$$\Pi_{\mathcal{X}}\left(W_\epsilon(f_0)^c \mid y^n\right) \to 0 \text{ a.s.}$$

Strong consistency:

$$\Pi_{\mathcal{X}}\left(S_\epsilon(f_0)^c \mid y^n\right) \to 0 \text{ a.s.}$$

$$W_\epsilon(f_0) = \left\{f : f \in \mathcal{F}, \left|\int_{\Re}\{g(x)f(x) - g(x)f_0(x)\}dx\right| < \epsilon, g : \Re \to \Re \text{ bounded continuous}\right\}$$

$$S_\epsilon(f_0) = \left\{f : f \in \mathcal{F}_d, \|f - f_0\|_1 < \epsilon\right\}$$

Posterior consistency

- Basic idea: posterior probability of an arbitrary nbd around f_0 goes to 1 as $n \to \infty$
- Weak consistency: $\Pi(W_\epsilon(f_0) \mid y^n) \to 1$ a.s. f_0
- Strong consistency: $\Pi(S_\epsilon(f_0) \mid y^n) \to 1$ a.s. f_0
- Early result by Doob (1948): posterior consistent a.e. on prior support, not useful to check consistency at a particular density
- Breakthrough result by Schwartz (1965)

Consistent tests

- Let $f_0 \in \mathcal{F}$ and U be some nbd of f_0
- Intuitively, should be able to separate f_0 from U^c - formalized through consistent tests
- A test function $\phi_n(y^n)$ is a non-negative measurable function bounded by 1
- Suppose testing $H_0 : f = f_0$ vs $H_1 : f \in U^c$
- $\phi_n(y^n)$ can be thought of as a randomized decision rule so that $\phi_n(y^n) = I(\text{Rejection region}|y^n)$
- A sequence of test functions said to be uniformly consistent if both probabilities of type I and II errors converge to 0 as n increases

Exponentially consistent & unbiased tests

- $\{\phi_n(y^n)\}$ is uniformly exponentially consistent if there exist constants $C, \beta > 0$ such that

$$\mathrm{E}_{f_0}[\phi_n(y^n)] \leq C \exp(-n\beta)$$
$$\sup_{f \in U^c} [1 - \phi_n(y^n)] \leq C \exp(-n\beta)$$

- $\{\phi_n(y^n)\}$ is strictly unbiased if

$$\mathrm{E}_{f_0}[\phi_n(y^n)] < \inf_{f \in U^c} [\phi_n(y^n)]$$

- The two notions above are equivalent (Hoeffding's inequality)
- Unbiased tests often easier to construct

Schwartz's theorem

Theorem

Let Π be a prior on $\mathcal{F}$ and $f_0 \in \mathrm{KL}(\Pi)$. If there exist a sequence of exponentially consistent tests for $H_0 : f = f_0$ vs $H_1 : f \in U^c$, then $\Pi(U \mid y^n) \to 1$ a.s. $P_{f_0}^{\infty}$

- Note $f_0 \in \mathrm{KL}(\Pi)$ means for any $\epsilon > 0$, $\Pi(KL_\epsilon(f_0)) > 0$
- Loosely speaking, Schwartz's theorem states large KL support $+$ model identifiability condition $\implies$ posterior consistency
- The KL distance related to likelihood ratios, since $(1/n) \sum_{i=1}^{n} \log\{f_0(Y_i)/f(Y_i)\} \to KL(f_0, f)$ by SLLN

Why Schwartz' s theorem works?

Exp. Cons. sequence of tests for $f = f_0$ vs. $f \in U^c$ makes likelihood ratio small

$$\Pi(U^c \mid \mathbf{Y}^n) = \frac{\int_{U^c} \prod_{i=1}^n \frac{f(Y_i)}{f_0(Y_i)} \Pi(df)}{\int_{\mathcal{F}} \prod_{i=1}^n \frac{f(Y_i)}{f_0(Y_i)} \Pi(df)} \quad \begin{array}{l} \longrightarrow N_n \\ \longrightarrow D_n \end{array}$$

$$\int_{\mathcal{F}} e^{-\sum_{i=1}^n \log \frac{f_0(Y_i)}{f(Y_i)}} \Pi(df) \rightarrow \int_{\mathcal{F}} e^{-nKL(f_0;f)} \Pi(df)$$

$$f_0 \in KL(\Pi) \Rightarrow \liminf e^{n\beta} D_n = \infty, \ \forall \beta > 0.$$

Specialized conditions for weak and strong consistency

- Turns out that the exponentially consistent test criterion is difficult to verify
- Need easy to verifiable conditions specific to neighborhoods

Theorem: weak

If $f_0 \in \mathrm{KL}(\Pi)$, the posterior is weakly consistent at f_0.

Specialized conditions for weak and strong consistency

- Turns out that the exponentially consistent test criterion is difficult to verify
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Theorem: weak

If $f_0 \in \mathrm{KL}(\Pi)$, the posterior is weakly consistent at f_0.

Theorem: strong (Ghosal et al. 1999)

If $f_0 \in \mathrm{KL}(\Pi)$ and there exists a sequence of subsets $\mathcal{F}_n \subset \mathcal{F}$ such that for any $\epsilon > 0$

1. $\log N(\epsilon, \mathcal{F}_n, || \cdot ||_1) \approx o(n)$
2. $\Pi(\mathcal{F}_n^c) \leq e^{-cn}$

then the posterior is L_1-consistent at f_0.

Weak consistency

Weak consistency: If U_ϕ is a weak neighborhood of f_0, for a bounded conts. function ϕ

$$U_\phi = \left\{ f : \left| \int \phi f - \int \phi f_0 \right| < \epsilon \right\}$$

Choose the test function to be ϕ since

Type I error: $E_{f_0}\{\phi(Y_1)\} = \int \phi f_0$ and

Power: $\inf_{f \in U_\phi^c} \int \phi f \geq \int \phi f_0 + \epsilon$

$\Rightarrow$ existence of unbiased sequence of tests

KL condition suffices for weak consistency

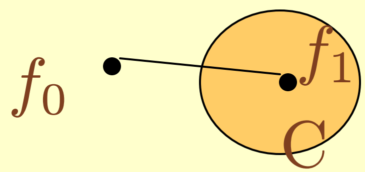
Strong consistency – Why Ghosal et al. 1999 works?

Strong consistency: If U is a strong nhbd. of f_0 i.e.

$$U = \{f : \|f - f_0\|_1 < \epsilon\}$$

Trivial to construct exponential consistent tests for

$$H_0 : f = f_0 \ \& \ H_1 : f \in C$$



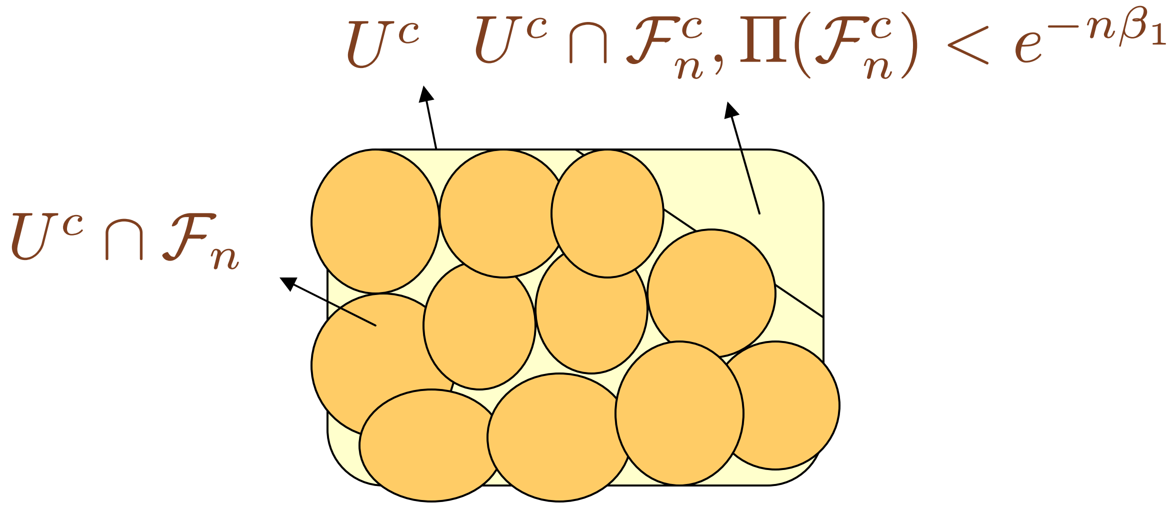
How do we do it?

$$C = \{f : \|f - f_1\|_1 \leq \|f_1 - f_0\|_1 /2\}$$

Take $B = \{y : f_1(y) > f_0(y)\}$ and $\Phi = I_B$

Then $E_{f_1}(\Phi) \geq E_{f_0}(\Phi) + \|f_1 - f_0\|_1 /2$

Why Ghosal et al 1999 works?



$$N_n \leq \int_{U^c \cap \mathcal{F}_n} \prod_{i=1}^n \frac{f(Y_i)}{f_0(Y_i)} \Pi(df) + \int_{\mathcal{F}_n^c} \prod_{i=1}^n \frac{f(Y_i)}{f_0(Y_i)} \Pi(df)$$

T_{1n} T_{2n}

$$T_{1n} \leq N(\epsilon, \|\cdot\|, \mathcal{F}_n) e^{-n\beta_{01}}, \; T_{2n}/D_n < e^{-n\beta_{02}}$$

For L_1 consistency, we need

$$\Pi(\mathcal{F}_n^c) < e^{-n\beta_1},$$

$$\log N(\epsilon, \|\cdot\|, \mathcal{F}_n) \approx o(n)$$

Example: Density estimation using DPM

- $Y_1, Y_2, \ldots, \sim f_0 \in \mathcal{F}$, want to estimate f_0
- We specify Π by $Y_i \sim N(\mu_i, \sigma_i^2), (\mu_i, \sigma_i^2) \mid \mathcal{P} \sim \mathcal{P}, \mathcal{P} \sim DP(\alpha G_0)$, G_0 a distribution on $\Re \times \Re^+$, π_h are constructed by stick-breaking $Beta(1, \alpha)$ variates.
- Induced density of $Y_i, f(y_i) = \sum_{h=1}^{\infty} \pi_h N(y_i, \mu_h, \sigma_h^2), (\mu_h, \sigma_h)^2 \sim G_0$
- Under what conditions on f_0 and G_0 do we have weak and strong posterior consistency?

Weak cons. in DPM (Ghosal et al. 1999; Tokdar 2006)

Split $KL(f_0, f)$ into 2 parts

$$KL(f_0, f) = \int_{\Re} f_0(y) \log \frac{f_0(y)}{\tilde{f}(y)} dy + \int_{\Re} f_0(y) \log \frac{\tilde{f}(y)}{f(y)} dy.$$

$$\tilde{f} = \int \phi(\tfrac{y-\mu}{\sigma}) \tilde{G}(\mu, \sigma)$$

$$f = \int \phi(\tfrac{y-\mu}{\sigma}) G(\mu, \sigma)$$

T_1

Impose tail conditions on f_0 to approximate f_0 by compactly supported $\tilde{G}$ needed to construct density $\tilde{f}$

T_2

Verify $\tilde{f}$ in the weak support of Π and also T_2 is arbitrarily small

Constructing $\tilde{f}$: approximation idea

Know $\int_{\Re} \frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right) f_0(\mu) d\mu \to f_0(y)$ as $\sigma \to 0$

↓

Want compactly supported $\tilde{G}_n$ s.t.
$\int_{\Re} \frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right) d\tilde{G}_n(\mu,\sigma) \to f_0(y)$

↓

$d\tilde{G}_n(\mu,\sigma) \propto \delta_{\sigma_n}(\sigma) f_0(\mu) I_{[-n,n]}(\mu)$
with $\sigma_n \to 0$

Handling T_1

A1. f_0 is nowhere zero, continuous and bounded by $M < \infty$.
A2. $|\int_{\Re} f_0(y) \log f_0(y) dy| < \infty$.
A3. $|\int_{\Re} f_0(y) \log \frac{f_0(y)}{\psi(y)} dy| < \infty$,
where $\psi(y) = \inf_{t \in [y-1, y+1]} f_0(t)$.
A4. $\exists\ \eta > 0$ such that $\int_{\mathcal{Y}} |y|^{2(1+\eta)} f_0(y) dy < \infty$.

Under (A1)-(A4), using a compactly supported sequence $\tilde{G}_n$,

$$f_n(y) = \int \frac{1}{\sigma} \phi\left(\frac{y-\mu}{\sigma}\right) d\tilde{G}_n(\mu, \sigma)$$

approximates $f_0(y)$ and makes T_1 arbitrarily small as $n \to \infty$. Choose $\tilde{f} = f_{n_0}$ for large enough n_0.

Handling T_2

Find a weak nhbd W of $\tilde{G}_{n_0}$ such that for $G \in W$, T_2 is small.

$$T_2 = \int_{\Re} f_0(y) \log \left\{ \frac{\phi * \tilde{G}_{n_0}(y)}{\phi * G(y)} \right\} dy < \epsilon$$

$$\int_{|y| \le k} \cdot$$

$$W = \{G : \left| \phi * \tilde{G}_{n_0}(y_i) - \phi * G(y_i) \right| < \epsilon\}$$

$$\int_{|y| \ge k} \cdot$$

use tail condition

What are the pieces left ?

Need to ensure that a DP assigns some mass at W

TRUE if G_0 has full support

Strong consistency in DPM (Sieve construction)

How do we construct a sieve $\mathcal{F}_n$ such that

1. $\log N(\mathcal{F}_n, \|\cdot\|, \epsilon) = o(n)$
2. $\Pi(\mathcal{F}_n^c) = O(e^{-n})$

Ghosal et al. 1999 restrictive in terms of applicability

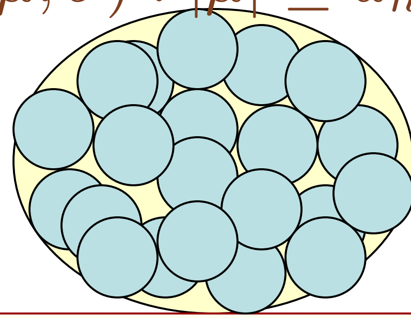
An alternative (Pati, Dunson & Tokdar, 2011)

$\mathcal{F}_n$ resembles finite mixtures $\sum_{h=1}^{m_n} \pi_h \frac{1}{\sigma_h} \phi\left(\frac{y-\mu_h}{\sigma_h}\right)$

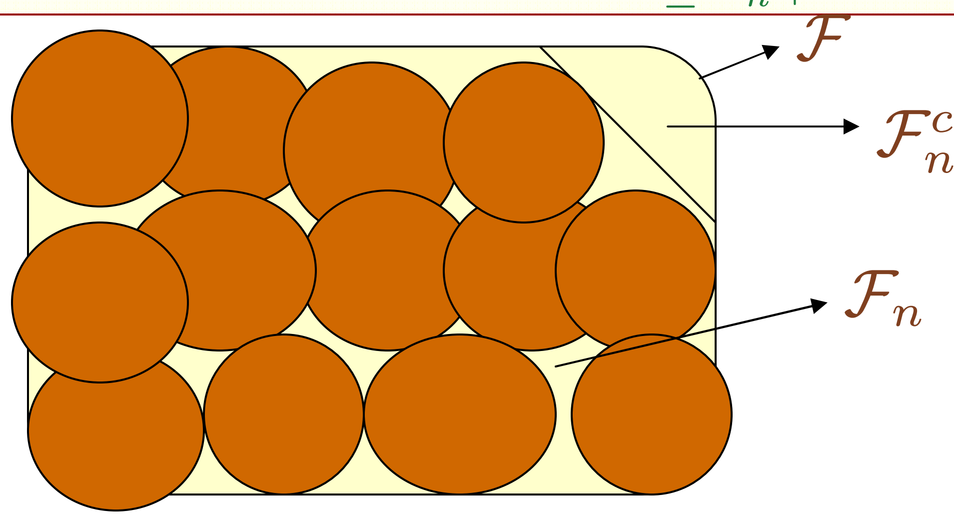
1. First few atoms are in a compact set
2. Tail sum is small

Strong consistency in DPM (Sieve construction)

$$\Theta_{a_n,h_n,l_n} = \{(\mu,\sigma) : |\mu| \le a_n, l_n \le \sigma \le h_n\}.$$



$$\mathcal{F}_n = \left\{ f : f(y) = \sum_{h=1}^{\infty} \pi_h \frac{1}{\sigma_h} \phi\left(\frac{y-\mu_h}{\sigma_h}\right), \{(\mu_h,\sigma_h)\}_{h=1}^{m_n} \in \Theta_{a_n,h_n,l_n}, \sum_{h \ge m_n+1} \pi_h \le \epsilon \right\}.$$



Sieve construction (Contd.)

For $f_1, f_2 \in \mathcal{F}_n, \|f_1 - f_2\|_1 \leq$

$$\int_{\mathcal{X}} \sum_{h=1}^{m_n} \pi_h^{(1)} \left| \phi_{\mu_h^{(1)}, \sigma_h^{(1)}}(y) - \phi_{\mu_h^{(2)}, \sigma_h^{(2)}}(y) \right| dy$$

$$+ \sum_{h=1}^{m_n} \left| \pi_h^{(1)} - \pi_h^{(2)} \right| + 2\epsilon.$$

balls needed$= N(\Theta_{a_n, h_n, l_n}, \epsilon, \|\cdot\|) \leq d_1 \left(\frac{a_n}{l_n}\right) + d_2 \log \frac{h_n}{l_n} + 1.$

balls needed$= N(\mathcal{F}_n, 4\epsilon, \|\cdot\|_1) \leq$

$$\left\{ d_1 \left(\frac{a_n}{l_n}\right) + d_2 \log \frac{h_n}{l_n} + 1 \right\}^{m_n} m_n^{m_n}$$

Strong consistency (Choice of m_n, a_n, l_n, h_n)

1. If $G_0 = N_p(\mu; \mu_0, \Sigma_0) \times IG(\sigma^2; a, b)$, then $a_n = O(\sqrt{n}), l_n = O(\frac{1}{\sqrt{n}}), h_n = e^n$.
2. $P(\sum_{h=m_n+1}^{\infty} \pi_h > \epsilon) \leq e^{-m_n \log m_n}$, $m_n = O\left(\frac{n}{\log n}\right)$
3. With these choices of m_n, a_n, l_n, h_n, given any $\xi > 0$,

$$\log(N(\mathcal{F}_n, 4\epsilon, \|\cdot\|_1)) = o(n),$$

4. $\Pi(\mathcal{F}_n^c) \leq O(e^{-n})$

Posterior convergence rates

- Once we have consistency, natural to ask whether we can characterize how fast the posterior concentrates
- In posterior consistency, we consider a fixed ball of radius ϵ around f_0
- Let the ball around f_0 shrink with n as fast as possible so that it still captures most of the posterior mass
- The minimum possible such sequence ϵ_n such that $E\{\Pi(f : d(f, f_0) \geq M\epsilon_n \mid y^n\} \to 0$ is called the rate of convergence of the posterior

Main rate theorem

Ghosal, Ghosh & van der Vaart (2000)

Suppose that for a sequence $\epsilon_n \to 0$ with $n\epsilon_n^2 \to \infty$, a constant $C > 0$ and sets $\mathcal{F}_n \subset \mathcal{F}$, one has

$$\log N(\epsilon_n, \mathcal{F}_n, d) \le C_1 n\epsilon_n^2$$

$$\Pi(\mathcal{F}_n^c) \le C_3 \exp\{-n\epsilon_n^2(C_2 + 4)\}$$

$$\Pi\left(f_{\mu,\sigma} : \int f_0 \log \frac{f_0}{f_{\mu,\sigma}} \le \epsilon_n^2, \int f_0 \log\left(\frac{f_0}{f_{\mu,\sigma}}\right)^2 \le \epsilon_n^2\right) \ge C_4 \exp\{-C_2 n\epsilon_n^2\}.$$

Then, for sufficiently large M, $\mathrm{E}\{\Pi(f : d(f, f_0) \ge M\epsilon_n \mid y^n\} \to 0$

- A more subtle interplay, roughly requires prior to be uniformly spread over the parameter space
- d usually Hellinger or L_1 metric

Application to a specific problem

- Density estimation model (Kundu & Dunson, 2011)

$$y_i = \mu(\eta_i) + \epsilon_i,\ \eta_i \sim \mathrm{U}(0,1),$$
$$\epsilon_i \sim \mathrm{N}(0,\sigma^2),\ (i = 1,\ldots,n).$$

- f_0 true density, F_0 c.d.f. with $\mu_0 = F_0^{-1} : (0,1) \to \Re$, induced density $f_{\mu_0,\sigma}(y) =$

$$\int_0^1 \phi_\sigma(y - F_0^{-1}(t))dt = \int_{a_0}^{b_0} \phi_\sigma(y-z)f_0(z)dz$$

- $f_{\mu_0,\sigma}(y) = \phi_\sigma * f_0(y)$, smoothness assumptions on f_0 imply $d(f_0, f_{\mu_0,\sigma}) \to 0$ as $\sigma \to 0$
- f_0 compactly supported implies $\mu_0 : [0,1] \to [a_0, b_0]$
- f_0 supported on $\Re$ implies $|\mu_0(t)| \to \infty$ as $t \to 0/1$

Prior specification

- Prior for $(\mu, \sigma) \in C([0,1]) \otimes (0, \infty)$ induces a prior on the space of densities on $(\Re, \mathcal{B})$
- Intuition: Π_μ concentrating around μ_0 and Π_σ around zero would imply $f_{\mu,\sigma}$ places +ve probability to arbitrary nbds of f_0
- Induced measure $\nu_\mu(B) = \tilde{\lambda}(\mu^{-1}(B))$, $\mu : ([0,1], \tilde{\lambda}) \to (\Re, \mathcal{B})$ m'ble, $\tilde{\lambda}$ Leb. meas. on $[0,1]$
- Marginalizing out η_i, induced density $f_{\mu,\sigma}$,

$$f_{\mu,\sigma}(y) = \int_0^1 \phi_\sigma(y - \mu(t))dt = \int \phi_\sigma(y - z)\nu_\mu(dz)$$

Review of Gaussian processes

- Want mechanism to produce random (continuous) functions.
- A random vector $X : (\Omega, \mathcal{E}, P) \to \Re^k$ is Gaussian if $a^{\mathrm{T}} X$ is Gaussian for any $a \in \Re^k$
- Let $X : (\Omega, \mathcal{E}, P) \to (\mathcal{C}[0,1], || \cdot ||_\infty)$ be measurable
- X is called <u>Gaussian</u> if $L(X)$ is Gaussian for any linear functional L
- For example, $L(f) = f(1/2)$, $L(f) = 2f(1/3) - f(3/4)$, ...
- Clearly, for any $(t_1, \dots, t_m)$, $\sum_{i=1}^m a_i X(t_i)$ is Gaussian for any $a \in \Re^m$
- $(X_{t_1}, \dots, X_{t_m})$ is MVN

Covariance kernel approach

- Specify a joint Gaussian for $(X_{t_1}, \ldots, X_{t_m})$ consistently
- Let $C(t,s)$ be a positive definite covariance kernel, i.e., $\mathbf{C} = \big(C(t_i, t_j)\big)$ is positive definite for any $t_1, \ldots, t_m$
- $(X_{t_1}, \ldots, X_{t_m}) \sim N(0, \mathbf{C})$, so that $C(s,t) = \text{cov}(X_s, X_t)$
- Common examples: $C(t,s) = \min(t,s)$, $C(t,s) = \exp(-\kappa|t-s|)$, $C(t,s) = \exp(-\kappa|t-s|^2)$ etc

Series expansion approach

- Mercer's theorem: There exists a sequence of eigenvalues $\lambda_h \downarrow 0$ and an orthonormal system of eigenfunctions ϕ_h, such that

$$C(s,t) = \sum_{h=1}^{\infty} \lambda_h \phi_h(s) \phi_h(t)$$

- Define $\tilde{X}(t) = \sum_{h=1}^{\infty} \lambda_h^{1/2} \, Z_h \, \phi_h(t)$, where Z_h i.i.d. $\mathrm{N}(0,1)$
- $\mathrm{cov}\left(\tilde{X}_s, \tilde{X}_t\right) = \sum_{h=1}^{\infty} \lambda_h \phi_h(s) \phi_h(t) = C(s,t)$
- We can start with a series representation by choosing λ_h and ϕ_h. Different choices lead to splines, neural networks, wavelets, etc

RKHS of Gaussian processes

- In np Bayes, want priors to place positive probability around arbitrary neighborhoods of a large class of parameter values (large support property)
- The prior concentration plays a key role in determining the rate of posterior contraction
- The reproducing kernel Hilbert space (RKHS) of a Gaussian process determines the prior support and concentration
- Let X be a zero mean Gaussian process on $[0, 1]$ with covariance kernel $C(s, t) = E(X_s X_t)$
- The RKHS $\mathbb{H}$ is the completion of the linear space

$$f(t) = \sum_{h=1}^{m} a_h C(s_h, t),\ s_h \in [0, 1],\ a_h \in \Re.$$

- Intuitively, a space of functions that are similar to the covariance kernel in terms of smoothness

Properties

- If $f_1(t) = C(s_1, t), f_2(t) = C(s_2, t)$, define $(f_1, f_2)_{\mathbb{H}} = C(s_1, s_2)$. Extend linearly and continuously to whole of $\mathbb{H}$
- Finite-dimensional case: let $X \sim N_k(0, \Sigma)$, Σ pd. Then $\mathbb{H} = \Re^k$, $(x, y)_{\mathbb{H}} = x^{\mathrm{T}}\Sigma^{-1}y$ and hence $||x||^2_{\mathbb{H}} = x^{\mathrm{T}}\Sigma^{-1}x$. Same RKHS norm on density contours!!
- The support of a mean zero Gaussian process is the closure of the RKHS. For many standard covariance kernels, the support equals $\mathcal{C}[0, 1]$
- The rate of posterior contraction at a function f_0 depends on

$$\phi_{f_0}(\epsilon) = \inf_{h \in \mathbb{H}: ||h - f_0||_{\mathbb{H}} < \epsilon} ||h||^2_{\mathbb{H}} - \log \Pr(||X||_\infty < \epsilon)$$

Back to the rates problem

- Ongoing work (Pati, Bhattacharya & Dunson, 2011) on posterior convergence rates in NL-LVM model
- Only focus on the compactly supported case here
- Analysis of non-compact case more involved as quantile function of a non-compact density not in $C[0,1]$
- Standard sieve available for GP priors (van der Vaart & van Zanten 2007 onwards) - clever application of Borel's inequality
- KL condition main hurdle

Details

- Assume f_0 twice continuously differentiable, optimal minimax rate in that case $n^{-2/5}$
- Using a GP prior with squared exponential covariance kernel for μ & an inverse-gamma prior for σ, we achieve the minimax rate up to a log-factor
- One has

$$\int f_0 \log \left(\frac{f_0}{f_{\mu,\sigma}} \right)^2 \leq h^2(f_0, f_{\mu,\sigma}) \left(1 + \log || \frac{f_0}{f_{\mu,\sigma}} ||_\infty \right)^2$$

- With $\epsilon_n = n^{-2/5} (\log n)^\kappa$ and $\sigma_n^4 = \epsilon_n^2$,

$$\{\sigma \in [\sigma_n, \sigma_n + \sigma_n^b], ||\mu - \mu_0||_\infty \precsim O(\sigma_n^3)\} \subset \left\{ \int f_0 \log \frac{f_0}{f_{\mu,\sigma}} \precsim \sigma_n^4, \int f_0 \log \left(\frac{f_0}{f_{\mu,\sigma}} \right)^2 \precsim \sigma_n^4 \right\}.$$

Appendix - list of common inequalities

- $||p - q||_1^2 \leq 4h^2(p, q) \leq 4||p - q||_1$
- $\mathrm{KL}(p, q) \geq ||p - q||_1^2/2$
- $\mathrm{KL}(p, q) \leq h^2(p, q)\{1 + \log ||p/q||_\infty\}$
- $p = \mathrm{N}(\mu_1, \sigma_1^2), q = \mathrm{N}(\mu_2, \sigma_2^2)$ with $\sigma_2 > \sigma_1 > \sigma_2/2$, then $||p - q||_1 \leq (2/\pi)^{0.5}|\mu_1 - \mu_2|/\sigma_2 + 3(\sigma_2 - \sigma_1)/\sigma_1$

Key references:

- Ghosal's research page
  http : //www4.stat.ncsu.edu/ sghosal/papers.html
- van der Vaart's page
  http : //www.few.vu.nl/ aad/research.html
- van Zanten's page
  http : //www.win.tue.nl/ jzanten/research.html
- Key consistency references: Barron, Schervish & Wasserman, 1999; Ghosal, Ghosh & Ramamurthy, 1999; Tokdar, 2006; Tokdar & Ghosh, 2007
- Key rates references: Ghosal, Ghosh & van der Vaart, 2000; Ghosal & van der Vaart, 2001; Ghosal & van der Vaart, 2007; van der Vaart & van Zanten, 2007-2009.
- Several others not cited ! Our apologies - See references within these articles.