Eig of Graphs

Let A be the adjacency matrix for an undirected graph with no self loops.

$$A \in \mathbb{R}_+^{n \times n} \qquad A_{ij} \geq 0$$

Def: $d = A\mathbb{1}$ where $\mathbb{1}$ is a vector of ones.

$d_i = \sum_j A_{ij}$ This is the "degree" of node i

$D \in \mathbb{R}^{n \times n}$ D is diagonal $D_{ii} = d_i \;\forall i$.

Graph Laplacian

$$L_{un} = D - A$$

$\boxed{\text{Facts}}$ Let $f \in \mathbb{R}^n$ be a "function on the nodes"

↳ for node i $f_i \in \mathbb{R}$

$$f' L_{un} f = \sum_{ij} f_i f_j [L_{un}]_{ij}$$

$$= \frac{1}{2} \sum_{ij} A_{ij} (f_i - f_j)^2$$

$\boxed{\text{fact 1}}$

Pf $f' L_{un} f = f'(D-A) f = f'Df - \underline{f'Af}$

$$= \underbrace{\sum_{i=1}^{n} d_i f_i^2} - \sum_{ij} f_i f_j A_{ij}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} d_i f_i^2 - 2 \sum_{ij} f_i f_j A_{ij} + \underbrace{\sum_{j=1}^{n} d_j f_j^2}_{A \text{ is sym.}} \right]$$

$$= \frac{1}{2}\left[\underline{\sum_{i=1}^{n} \sum_{j=1}^{n}} \underline{A_{ij}} f_i^2 - 2 \underline{\sum_{ij}} f_i f_j \underline{A_{ij}} + \underline{\sum_{j=1}^{n} \sum_{i=1}^{n}} \underline{A_{ij}} f_j^2 \right]$$

$$= \frac{1}{2} \sum_{ij} A_{ij} \left(f_i^2 - 2 f_i f_j + f_j^2 \right)$$

$$= \frac{1}{2} \sum_{ij} A_{ij} \left(f_i - f_j \right)^2$$

Fact 1 $\quad f' L_{un} f = \frac{1}{2} \sum_{ij} A_{ij} (f_i - f_j)^2$

Fact 2 $\quad L_{un}$ is pos semi-definite & sym.

pf $\quad f' L_{un} f \geq 0$ because $A_{ij} \geq 0$

$\boxed{D - A}$ both sym. $\qquad (f_i - f_j)^2 \geq 0$.

Fact 3 $\quad \mathbb{1}$ is an eigenvector with eigval zero.

$$\underline{L_{un} \mathbb{1}} = (D - A)\mathbb{1} = D\mathbb{1} - A\mathbb{1} = d - d = 0 \cdot \mathbb{1}$$

$$\underline{f' L_{un} f} = \frac{1}{2} \sum A_{ij} \underline{(f_i - f_j)}^2$$

smallest eigenvector of Lun solves $\quad \min_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} x^T L_{un} x$

Suppose $A_{ij} \in \{0, 1\} \qquad \underline{A_{ij} = 1} \iff \underline{i \sim j}$

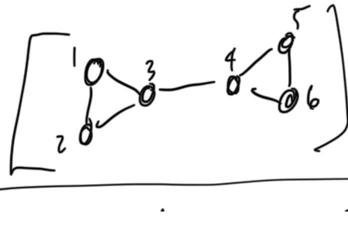
$$= \frac{1}{2} \sum_{i \sim j} (f_i - f_j)^2$$

second smallest eigenvector solves

$$\min_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \frac{1}{2} \sum_{i \sim j} (x_i - x_j)^2$$

& $\langle x, \mathbb{1} \rangle = 0$

Exercise: Find 2^{nd} smallest eigenvector of Lun for



Inspect the values on the nodes.

Fact 4

3 connected components

$S_1 = \{1, 2, 3\}$
$S_2 = \{4, 5\}$
$S_3 = \{6\}$

$$\mathbb{1}(S_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \qquad \mathbb{1}(S_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Suppose the graph has k connected components. Then the multiplicity of the eigenvalue zero is k. Let $S_1, \dots, S_k \subset \{1, \dots, n\}$ be the sets of nodes in the connected components.

Def $\mathbb{1}(S_j) \in \mathbb{R}^n$ $\quad \left[\mathbb{1}(S_j)\right]_i = \begin{cases} 1 & i \in S_j \\ 0 & \text{o.w.} \end{cases}$

These $\underline{\mathbb{1}(S_1), \dots, \mathbb{1}(S_k)}$ are all eigenvectors with eigenvalue zero.

pf Let $f = \mathbb{1}(S_j)$

$$f' L_{un} f = \frac{1}{2} \sum_{i \sim j} (f_i - f_j)^2 = 0$$

If $i \sim j$ then $f_i = f_j$!

Example: Suppose 1 connected component.

$\mathbb{1}(S_1) = \mathbb{1} \in \mathbb{R}^n$ see fact 3.

One Note $\mathbb{1}(S_j)$ indicate cluster membership!

Problem $\quad x = \sum \alpha_j \underline{\mathbb{1}(S_j)} \quad$ x is also eigenvector with eigval zero!

These might be the eigenvector your computer gives you. So, you need another processing step to recover the "clusters".

Before $\quad L_{un} = D - A.$

$$L = I - D^{-1/2} A D^{-1/2} = \boxed{D^{-1/2} L_{un} D^{-1/2}}$$

$$= I - D^{-1} A = D^{-1} L_{un}$$

$L_{RW} = I - D^{-1}A$

Fact 1: $f \in \mathbb{R}^n$

$$f' L f = \frac{1}{2} \sum_{ij} A_{ij} \left(\frac{f_i}{\sqrt{d_i}} - \frac{f_j}{\sqrt{d_j}} \right)^2$$

Fact 2: $Lf = \lambda f \iff L_{RW} \tilde{f} = \lambda \tilde{f}$ where $\tilde{f} = D^{1/2} f$.

pf $D^{-1/2} L_{un} D^{-1/2} f = \lambda f$

Left multiply by $D^{-1/2}$

$$D^{-1/2}\left[D^{-1/2} L_{un} \underbrace{D^{-1/2} f}\right] = \lambda \underbrace{D^{-1/2} f}$$

$$\underbrace{D^{-1} L_{un}} \tilde{f} = \lambda \tilde{f}$$

$$L_{RW} \tilde{f} = \lambda \tilde{f}$$

Fact 3: $\mathbb{1} \in \mathbb{R}^n$ is an eigenvector of L_{RW} ~~as~~

& $D^{-1/2}\mathbb{1} \in \mathbb{R}^n$ is an eigenvector of L

Both have eigenvalue zero.

$$L_{RW}\mathbb{1} = (I - D^{-1}A)\mathbb{1} = \mathbb{1} - D^{-1}(A\mathbb{1}) = \mathbb{1} - \underbrace{D^{-1}d}_{\mathbb{1}} = \mathbb{1} -$$

$$\left[D^{-1}d\right]_i = \frac{d_i}{d_i} = 1 \quad \forall i$$

Same as fact 4 above... k connected components

$\Rightarrow k$ eigenvectors w/ eigenvalue

$\Rightarrow$ these k eigenvectors ~~are~~

are $\mathbb{1}(S_i)$ for L_{RW}

... to get it for L

& use fact ...

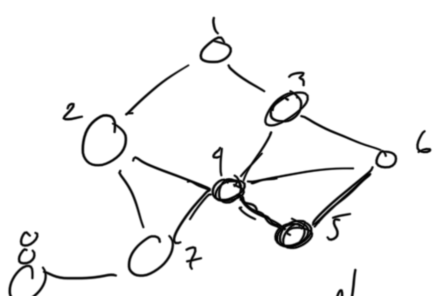
When I do data analysis...
I remove the "I -" ... just
use $D^{-1/2} A D^{-1/2}$. Then,
Question: How does this change eigenvectors & eigenvalues

Suggested due date for blog posts: Nov 10.

Why is it called Random Walk Laplacian?

$I - \underline{D^{-1}A}$

$P = D^{-1}A$. Markov transition Matrix on nodes of the graph corresponding to "Simple Random Walk"



on a graph

$$\boxed{\text{time series } X_t = X_{t-1} + N(0,1) \in \mathbb{R}}$$

Stochastic ~~proces~~ process indexed by $\mathbb{N}$ on $\mathbb{R}$.

$X_t \in \mathbb{R}$

$X_0 = 0$

X_t is a node.

X_t indexed by $\mathbb{N}$

X_t is a node in the graph

$X_0 = \{5\}$ $X_1 = 4$ $X_2 = 3$ ....

P is $n \times n$ where there are n states

$$P_{ij} = P(X_{t+1} = j \mid X_t = i) \qquad i \to j.$$

$$= \underbrace{\frac{A_{ij}}{d_i} \qquad = D^{-1}A.}_{D^{-1}A.}$$