# Markov Chains

cf Levin, Peres, Wilmer
Markov Chains and Mixing times

Often have a sequence of RV

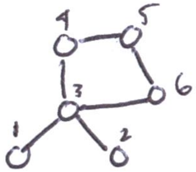
$X_0, X_1, \ldots, X_n$ (think over time)

Approximate the joint distribution w/ Markov assumption:

$$P(X_{t+1} \mid X_0, \ldots, X_t) = P(X_{t+1} \mid X_t)$$

on each step, the next state depends only on the current state.

## Random walk on G




$X_0 \sim$ initial distribution on $V$.

$X_{t+1} \mid X_t \sim \text{Uniform}(\text{Ne}(X_t))$
↑ neighbors.

## Markov transition matrix

Let $A \in \{0,1\}^{N \times N}$ be adjacency matrix. Def: $D \in \mathbb{Z}^{N \times N}$ diagonal

$$D_{ii} = \deg(i).$$

$$P = D^{-1} A.$$

$$P_{ij} = P(X_{t+1} = j \mid X_t = i)$$
$$= \frac{A_{ij}}{\deg(i)}.$$

Fact: $X_0 \sim \nu \in [0,1]^N$

$\Rightarrow X_t \sim \nu P^t$

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From now on in this discussion,

$G$ is undirected $\Rightarrow$ $A$ is symmetric

$\Rightarrow$ $P$ is "reversible"

i.e. $\exists\ \pi \in \mathbb{R}^N$ s.t.

$$\pi_i P_{ij} = \pi_j P_{ji}$$

Proof: $\pi_i = \deg(i)$

OR, to make it a prob dist

$$\pi_i = \frac{\deg(i)}{\sum_{j=1}^{N} \deg(j)}.$$

**Lemma:** $L_{sym} = D^{-1/2} A D^{-1/2}$ has same eigen values as $D^{-1}A$.

pf let $x, \lambda$ be an eigen pair of $L_{sym}$

$$Lx = \lambda x \Rightarrow D^{-1/2} A (D^{-1/2} x) = \lambda x$$

left mult by $D^{-1/2}$.

$$\Rightarrow D^{-1} A (D^{-1/2} x) = \lambda (D^{-1/2} x).$$

So, $D^{-1/2} x, \lambda$ is an eigen pair of $P$.

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$L_{sym}$ is symmetric. Let $\varphi_1, \dots, \varphi_N$ be eigen "functions".

$f_u \triangleq D^{1/2} \varphi_u$ are orthonormal w.r.

$$\langle f_u, f_v \rangle_\pi = \sum_{i \in V} f_u(i) f_v(i) \pi(i)$$

$$= \begin{cases} 0 & u \neq v \\ 1 & u = v. \end{cases}$$

Fact: $f_1$ can be the vector $\mathbb{1}$.

& $\varphi_1$ can be $\sqrt{\pi} = \mathrm{diag}(D^{1/2})$

Fact

$$P^t = D^{-1/2} L_{sym}^t D^{1/2}$$

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So, for any starting dist $\nu$, if $|\lambda_2| < 1$

$$\nu P^t = (\nu D^{-1/2}) \left( \sum \lambda_i^t \varphi_i \varphi_i^T \right) D^{1/2}.$$

$$\rightarrow \nu \underbrace{\cancel{D^{-1/2}}}_{\mathrm{diag}(\pi^{-1/2})} \lambda_1^t \sqrt{\pi} \sqrt{\pi}^T \mathrm{diag}(\cancel{\sqrt{\pi}})$$

$$= \nu \mathbb{1} \lambda_1^t \pi.$$