Eigenvalue Decomposition

Let $A^{\in \mathbb{R}^{N\times N}}$ be real & symmetric.

$\exists\ U \in \mathbb{R}^{N\times N}$ w/ $U^T U = I$

& $\Lambda \in \mathbb{R}^{N\times N}$ diagonal s.t.

$$A = U \Lambda U^T.$$

What happens when A is not symmetric?

↳ i.e. directed graph.

$\Rightarrow U$ & Λ are complex valued!

What do imaginary numbers mean? IDK.

The SVD is a "more natural" formulation.

Whenever anyone says this, they have no idea what they mean. *

Singular Value Decomposition (SVD)

Let $M \in \mathbb{R}^{N\times d}$ be real and potentially rectangular $\boxed{d \le N}$

"skinny SVD"

$\exists\ U \in \mathbb{R}^{N\times d}$ $\quad U^T U = I_d$

$V \in \mathbb{R}^{d\times d}$ $\quad V^T V = I_d$

$D \in \mathbb{R}^{d\times d}$ $\quad$ diagonal $\quad$ s.t.

D_{ii} in decreasing order ---

$D_{ii} \ge D_{i+1,i+1}\ \forall i$

$D_{ii} \ge 0\ \forall i$

$$M = U D V^T.$$

Fact 1: $d = N$ & M symmetric.

Eigen: $M = \tilde{U} \Lambda \tilde{U}^T$.

SVD: $M = U D V^T$

① $U = \tilde{U}$

② $V^T = \tilde{U}^T$ (If $\Lambda_{ii} \ge 0\ \forall i$)

Liow. $V^T = S\tilde{U}^T$

$S_{ii} = \text{sign}(\Lambda_{ii})$

For certain choice of $\tilde{U}$ if $\Lambda_{ii} = \Lambda_{jj}$ for some $i \ne j$.

③ $|\Lambda_{ii}| = D_{ii}$

Fact 1 shows that SVD generalized the "nice" eigen decomp to rectangular matrices.

* $\Rightarrow$ Who is to say that spectral clustering VO is not thresholding singular vectors? It would be equivalent.

Fact 2

$$\underset{\hat{M} \text{ is rank } k}{\arg\min} \sum_{ij} (M_{ij} - \hat{M}_{ij})^2 = \quad \| M - \hat{M} \|_F^2$$

$$= U_{:k} D_{:k} V_{:k}^T$$

$$= \sum_{i=1}^{k} U_i V_i^T D_{ii}$$

rank!

ith col of U.

$U_{:k}$ contains first k columns of U

sim for $D_{:k}$ & $V_{:k}$

Think PCA!

Computation.

Power method for eigenvectors,

$X_0 \in \mathbb{R}^{N \times k}$ initialize with iid $N(0,1)$ entries.

$A X_0 \to \tilde{X}_0$

$\tilde{X}_0 = \tilde{U}_0 D V^T$ (SVD)

$X_1 \leftarrow \tilde{U}_0 \in \mathbb{R}^{N \times k}$.

$A X_1 \to \tilde{X}_1$ . . . .

$X_t \to$ leading eigenvectors when A is symmetric & Real.

When A is sparse, this is superfast.

SVD

Fact 3 If

$$M M^T = \tilde{U} \tilde{\Lambda} \tilde{U}^T \quad \text{(eigen)}$$

$$M^T M = \tilde{V} \tilde{\tilde{\Lambda}} \tilde{V}^T \quad \text{(eigen)}$$

Then ① $\tilde{\Lambda} = \tilde{\tilde{\Lambda}}$

& ② $\tilde{U} \sqrt{\tilde{\Lambda}} \tilde{V}^T$ is the SVD of M.

square root element wise.

To compute leading singular vectors, Do power method with

$$M(M^T X)$$

when M sparse... Fast computation.

Fact 3 also leads to interpretation of SVD... $[A^T A]_{ij} = \ldots$

$[A A^T]_{ij} = \ldots$