CPMA 521, Fall 2001
Professor Larget
Assignment #2: due September 17, 2001

1. Inspired by the recently completed U.S. Open, consider problem 1.2. Instead of the problems asked, answer the following:

   (a) Find an expression in p and $q = 1 - p$ for the probability that the game first enters the final stage at Deuce/30–30.

   (b) Let a be the expected number of points remaining when the game is at Advantage B/30–40, b be the expected number of points remaining when the game is at Deuce/30–30, and c be the expected remaining number of points when the game is at Advantage A/40–30. Condition on the next point to find an expression for each of a, b, and c. (For example, $a = 1 \times q + (1 + b) \times p$. Solve these three equations in three unknowns for fixed p and q and then evaluate when $p = q = 1/2$.

   (c) Complete the S-PLUS program on the Web page to simulate the probability that A wins the game when $p = 0.6$. Your function should allow you to "play" the game

Solution:

(a) P(enter deuce) = P(A wins exactly 2 of the first four points) = $6p^2q^2$.

(b) By conditioning on the next point,

$$\begin{cases} a = q + p(1 + b) \\ b = q(1 + a) + p(1 + c) \\ c = q(1 + b) + p \end{cases}$$

These three linear equations with three unknowns may be solved by various methods. The solution is

$$a = \frac{1 + 2p^2}{1 - 2pq}, \qquad b = \frac{2}{1 - 2pq}, \qquad c = \frac{1 + 2q^2}{1 - 2pq}.$$

When $p = q = 1/2$, $b = 4$.

(c) S program:

```
# Program to simulate a tennis game.
# p is the probability that player A wins a point
# The first player to win at least four points
# and to lead by at least two points wins the game.
#
# Notice the syntax for "for loops" and "while loops"
# as well as the "list" object and the $ operator
# to extract an element from a list.

tennis <- function(p,ntimes=1)
```

```
{
   wins <- 0
#  aPoints <- rep(0,100)
   for(k in 1:ntimes) {
     x <- list(a=0,b=0)
     while(max(x$a,x$b)<4 || abs(x$a-x$b) < 2) {
       if(runif(1) < p)
         x$a <- x$a+1
       else
         x$b <- x$b + 1
     }
     if(x$a>x$b) {
       wins <- wins+1
#      aPoints[x$a] <- aPoints[x$a] + 1
     }
   }
   return(wins)
#  return(invisible(list(wins=wins,aPoints=aPoints)))
#$ this is only here to fix how emacs colors text of different type
}
```

Alternatively, a similar technique to part (b) can be used to find an algebraic expression for the probability that A wins. This is

$$\text{P(A wins the game)} = \frac{p^4(15 - 34p + 28p^2 - 8p^3)}{1 - 2p + 2p^2}$$

For $p = 0.6$, this is 0.7357.

2. A density function is $f(x) = cx^2(2 - x)$ for $0 < x < 2$.

   (a) Find the value of c.
   (b) Evaluate the probability that a random variable X from this distribution is less than 1.
   (c) Find the mean and variance of this distribution.
   (d) Use S-PLUS or R to graph the density of this function.

Solution:

(a) The density must integrate to one.

$$\int_0^2 cx^2(2 - x)\,dx = c\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{4}{3}c$$

so $c = 3/4$.

(b)

$$\mathsf{P}(X < 1) = \int_0^1 \frac{3}{4}x^2(2 - x)\,dx = \frac{3}{4}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{16}$$

(c)

$$E[X] = \int_0^2 \frac{3}{4} x^3 (2-x)\,dx = \frac{3}{4}\left(\frac{32}{4} - \frac{32}{5}\right) = \frac{6}{5}$$

$$E[X^2] = \int_0^2 \frac{3}{4} x^4 (2-x)\,dx = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{8}{5}$$

$$\mathrm{Var}[X] = \frac{8}{5} - \left(\frac{6}{5}\right)^2 = \frac{4}{25}$$

(d)
```
# function that calculates the density from problem 2

f <- function(x) {
  y <- rep(0,length(x))
  xx <- x[!(x<0 | x>2)]
  y[!(x<0 | x>2)] <- 3/4 * xx^2 * (2-xx)
  return(y)
}

# function to graph it

prob2 <- function() {
  u <- seq(-1,3,.01)
  y <- f(u)
  plot(u,y,type="l",xlab="x",ylab="density",main="f(x) = 3/4 x^2 (2-x)")
  abline(h=0,col=2)
  return(invisible())
}
```

3. Problem 1.45.

   Solution: $A \sim \mathrm{Unif}(0,1), B|A = x \sim \mathrm{Unif}(0,x)$.

$$\mathrm{E}(B) = \mathrm{E}(\mathrm{E}(B|A)) = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4}$$

4. Problem 1.42. Either solve analytically or write an S-PLUS or R program to simulate the problem and guess at the answer.

   Solution: Let $a = \mathrm{P(win|start)}$, $b = \mathrm{P(win|A\ won\ last)}$, $c = \mathrm{P(win|A\ out)}$, and $d = \mathrm{P(win|Opp\ won\ last)}$. Then,

$$\begin{cases} a = \frac{1}{2}\,b + \frac{1}{2}\,c \\ b = \frac{1}{2} + \frac{1}{2}\,c \\ c = \frac{1}{2}\,d \\ b = \frac{1}{2}\,b \end{cases}$$

   These equations have solution $(a, b, c, d) = (5/14, 4/7, 1/7, 2/7)$. Thus, A and B each win with probability 5/14 and C wins with probability 4/14.