CPMA 521, Fall 2001
Professor Larget
Assignment #3: due September 24, 2001

1. Do problem 2.2. You do not need to write out the entire probability transition matrix. It is sufficient to specify p_{ij} for $1 \le i, j \le N$. For example, you may say something like this.

$$p_{0j} = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$p_{ij} = \begin{cases} 0.5 & \text{if } i \ge 1 \text{ and } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

(a)

$$p_{ij} = \begin{cases} \frac{i}{N} \cdot (1-p) & \text{if } 1 \le i \le N, j = i-1 \\ \frac{i}{N} \cdot p + \frac{N-i}{N} \cdot (1-p) & \text{if } 0 \le i \le N, j = i \\ \frac{N-i}{N} \cdot p & \text{if } 0 \le i \le N-1, j = i+1 \end{cases}$$

(b)

$$p_{ij} = \begin{cases} \frac{N-i}{N} \cdot p & \text{if } 1 \le i \le N, j = i-1 \\ \frac{i}{N} \cdot (1-p) + \frac{N-i}{N} \cdot p & \text{if } 0 \le i \le N, j = i \\ \frac{i}{N} \cdot (1-p) & \text{if } 0 \le i \le N-1, j = i+1 \end{cases}$$

(c)

$$p_{ij} = \begin{cases} \frac{i}{N} \cdot \frac{N-i}{N} & \text{if } 1 \le i \le N, j = i-1 \\ \left(\frac{i}{N}\right)^2 + \left(\frac{N-i}{N}\right)^2 & \text{if } 0 \le i \le N, j = i \\ \frac{i}{N} \cdot \frac{N-i}{N} & \text{if } 0 \le i \le N-1, j = i+1 \end{cases}$$

2. Do problem 2.5.

Solution:

(a) X_n is the number of white balls after the nth black ball is drawn.

$$\begin{pmatrix} 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p & 0 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Y_n is the number of white balls after the nth ball is drawn.

$$\begin{pmatrix} 1-p & p & 0 & 0 & 0 \\ 0 & 1-\frac{3}{4}p & \frac{3}{4}p & 0 & 0 \\ 0 & 0 & 1-\frac{1}{2}p & \frac{1}{2}p & 0 \\ 0 & 0 & 0 & 1-\frac{1}{4}p & \frac{1}{4}p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Do problem 2.19.

   Solution: A binomial number of fish are scooped out at each pass.

$$p_{ij} = \binom{i}{i-j} p^{i-j}(1-p)^{i-j} \qquad \text{for } i \geq 0,\ 0 \leq j \leq i$$

4. Enter the matrix from the class example. Here is one way to do it.

```
r1 <- c(4/5,1/5,0,0)
r2 <- c(0,0,1/2,1/2)
r3 <- c(1/2,1/2,0,0)
r4 <- c(0,0,1/5,4/5)
P <- rbind(r1,r2,r3,r4)
```

   Now, repeatedly square this matrix until it does not appear to change. For example,

```
P2 <- P %*% P
P4 <- P2 %*% P2
P8 <- P4 %*% P4
```

   Guess the rational numbers that make up the limit of this matrix $P(n)$ as $n \to \infty$.

   Solution:

$$P^{\infty} = \begin{pmatrix} \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \end{pmatrix}$$

5. For the same matrix above, calculate the first several state distributions beginning with $\boldsymbol{\pi}^{(0)} = (1, 0, 0, 0)$. For example,

```
p0 <- c(1,0,0,0)
p <- p0
for(i in 1:20) {
    p <- p %*% P
    print(p)
}
```

   What do you guess is the limiting value of $\boldsymbol{\pi}^{(n)}$? How is related to what you discovered in the previous problem? Explain the connections between your answers.

   Solution:

$$p^{\infty} = \left(\frac{5}{14}, \frac{2}{14}, \frac{2}{14}, \frac{5}{14}\right)$$

   This is one row of the matrix above. The explanation is that the jth element of p^{∞} represents the probability of being in state j regardless of where the Markov chain begins. Thus, given information on the initial state should be irrelevant.