CPMA 521, Fall 2001
Professor Larget
Assignment #4: due October 1, 2001

1. Consider the Markov chain example of progress of labor in human birthing on page 62.
   (a) Enter the matrix into S-PLUS.
   (b) What is the probability that a woman in her first pregnancy who is 4 cm dilated will have given birth within six hours?
   (c) How much time after first being 4 cm dilated will 95% of all women in their first deliveries have given birth?

   Solution: (b) Six hours is twelve time steps. The 1,8 entry of P^{12} is 0.9225.

   (c) Exploration shows that $[P^{13}]_{1,8} = 0.948$ and $[P^{14}]_{1,8} = 0.966$, so seven hours are required to surpass 95%, although 6 and a half hours is closest.

2. Do problem 2.3.

   Solution: Ehrenfest urn model with $2N$ total balls, one ball picked randomly and switched at each time step. X_n is the number of balls in Box A at time n.

$$p_{ij} = \begin{cases} \frac{2N-i}{2N} & \text{if } 0 \le i \le 2N-1,\ j = i+1 \\ \frac{i}{2N} & \text{if } 1 \le i \le 2N,\ j = i-1 \end{cases}$$

3. Do problem 3.13.

   Solution: Let $\mu_{i,n} = \mathrm{E}(X_n | X_0 = i)$.

   (a) First, $\mathrm{E}(X_0 | X_0 = i) = i$. By conditioning on the state at time $n-1$ for time $n > 0$, we have the following.

$$\begin{aligned}
\mathrm{E}(X_n|X_0 = i) &= \sum_{j=0}^{2N} p_{ij}^{(n-1)}[\mathrm{E}(X_n|X_{n-1} = j)] \\
&= p_{i0}^{n-1} 1 + \sum_{j=1}^{2N-1} p_{ij}^{(n-1)} \left[(j+1)\frac{2N-j}{2N} + (j-1)\frac{j}{2N}\right] + p_{i,2N} 2N - 1 \\
&= p_{i0}^{n-1} 1 + \sum_{j=1}^{2N-1} p_{ij}^{(n-1)} \left[\left(1 - \frac{1}{N}\right) j + 1\right] + p_{i,2N} 2N - 1 \\
&= \sum_{j=0}^{2N} p_{ij}^{(n-1)} \left[\left(1 - \frac{1}{N}\right) j + 1\right] \\
&= 1 + \left(1 - \frac{1}{N}\right) \mu_{i,n-1}
\end{aligned}$$

(b) Show that $\mu_{i,n} = N + (i - N)(1 - 1/N)^n$. Proof by induction on n. Let $n = 0$. Then $\mu_{i,0} = N + (i - N)(1 - 1/N)^0 = N + (i - N) = i$, which holds. Now assume that $\mu_{i,n-1} = N + (i - N)(1 - 1/N)n - 1$.

$$\begin{aligned}
\mu_{i,n} &= 1 + (1 - 1/N)\mu_{i,n-1} \\
&= 1 + \big(1 - 1/n)(N + (i - N)(1 - 1/N)^{n-1}\big) \\
&= 1 + \big(1 - 1/n)(N + (i - N)(1 - 1/N)^{n-1}\big) \\
&= 1 + N + (i - N)(1 - 1/N)^{n-1} - 1 - (i - N)/N(1 - 1/N)^{n-1} \\
&= N + (i - N)(1 - 1/N)^n
\end{aligned}$$

By induction on n,

$$\mu_{i,n} = N + (i - N)(1 - 1/N)^n \qquad \text{for all } n \geq 0$$

(c)

$$\lim_{n \to \infty} N + (i - N)(1 - 1/N)^n = N$$

In the long run, each ball should spend half its time in each box, so it makes sense that on average there will be equal numbers in each box.

4. Suppose that a matrix $\mathbf{P}$ may be decomposed into

$$\mathbf{P} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$$

where $\mathbf{D}$ is a diagonal matrix and $\mathbf{U}^{-1}$ is the inverse matrix of $\mathbf{U}$, namely $\mathbf{U}^{-1}\mathbf{U} = \mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. (This decomposition is called the spectral decomposition of a matrix. Matrices which may be decomposed in this way are called *diagonalizable*. Matrices $\mathbf{P}$ and $\mathbf{D}$ in this example are *similar* matrices.)

Find an expression for $\mathbf{P}^n$.

Solution:

$$\begin{aligned}
\mathbf{P} &= \mathbf{U}\mathbf{D}\mathbf{U}^{-1} \\
\mathbf{P}^n &= \big(\mathbf{U}\mathbf{D}\mathbf{U}^{-1}\big)^n \\
&= \mathbf{U}\mathbf{D}(\mathbf{U}^{-1}\mathbf{U})\mathbf{D}(\mathbf{U}^{-1} \cdots \mathbf{U})\mathbf{D}\mathbf{U}^{-1} \\
&= \mathbf{U}\mathbf{D}^n\mathbf{U}^{-1}
\end{aligned}$$

5. Consider the transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

Use the function `eigen` in S-PLUS to decompose $\mathbf{P} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$ as in the previous problem.

```
> e <- eigen(P)
> U <- e$vectors
> Uinv <- solve(U)
> D <- diag(e$values)
```

(a) What is $\lim_{n\to\infty} \mathbf{D}^n$?

(b) What is the exact expression of $\lim_{n\to\infty} \mathbf{P}^n$?

(c) What is the exact expression of $\lim_{n\to\infty} \pi^{(0)}\mathbf{P}^n$ where $\pi^{(0)} = (1,0,0,0)$?

(d) Find a vector π such that $\pi\mathbf{P} = \pi$.

Solution:

(a)

$$\lim_{n\to\infty} \mathbf{D}^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} \mathbf{P}^\infty &= \lim_{n\to\infty} \mathbf{P}^n \\ \lim_{n\to\infty} \mathbf{U}\mathbf{D}^n\mathbf{U}^{-1} &= \left[\ \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4\ \right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \mathbf{v}_3^{\mathrm{T}} \\ \mathbf{v}_4^{\mathrm{T}} \end{bmatrix} \\ &= \mathbf{u}_1\mathbf{v}_1^{\mathrm{T}} \\ &= \begin{bmatrix} \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} \end{bmatrix} \end{aligned}$$

where $\mathbf{u}_i$ is the ith column of $\mathbf{U}$ and $\mathbf{v}_j^{\mathrm{T}}$ is the jth row of $\mathbf{U}^{-1}$.

(c)

$$\lim_{n\to\infty} \pi^{(0)}\mathbf{P}^n = (1,0,0,0)\mathbf{P}^\infty = (5/14, 2/14, 2/14, 5/14)$$

(d) $\pi = (5/14, 2/14, 2/14, 5/14)$

6. For the gambling problem in lecture where the initial stakes are $a = 1$ and $b = 5$ and the probability that A wins a single round is $p = 1/6$, find the expected number of times that A's fortune is two before the game ends. (You may solve this numerically or analytically.)

Solution: Let a_i be the expected number of times we hit state 2 before absorption (including the current time) give we are in state i. Then, by conditioning on the first round of the game, we have the following system of equations.

$$\begin{aligned}
a_1 &= \frac{1}{6}a_2 \\
a_2 &= \frac{5}{6}a_1 + \frac{1}{6}a_3 + 1 \\
a_3 &= \frac{5}{6}a_2 + \frac{1}{6}a_4 \\
a_4 &= \frac{5}{6}a_3 + \frac{1}{6}a_5 \\
a_5 &= \frac{5}{6}a_4
\end{aligned}$$

Numerically, this has solution $a_5 = 1.152$.

7. For the fish bowl problem, 2.19, classify each state as recurrent or transient when there are five fish in the bowl initially.

   Solution: The class $\{0\}$ is the only recurrent class. The class $\{1, 2, 3, 4, 5\}$ is transient.

8. (a) The integers $0, 1, 2, 3, 4$ are written in order around a circle. You begin at 0. At each time step, you are equally likely to move clockwise or counter-clockwise one position. Find the period of each recurrent class.

   (b) The integers $0, 1, 2, 3, 4, 5$ are written in order around a circle. You begin at 0. At each time step, you are equally likely to move clockwise or counter-clockwise one position. Find the period of each recurrent class.

   Solution: There is a single recurrent class in each part. For (a), the periodicity is 1, for (b) it is 2. In (a), after five steps it is possible to be anywhere. In (b), the state alternates between even and odd states.