CPMA 521, Fall 2001
Professor Larget
Assignment #5: due October 8, 2001

I expect that you will use S-PLUS or R for many or all of these problems.

1. Consider another version of the game Paul and I played in class. Paul begins with a chips and I begin with one chip. At each stage, Paul wins one chip from me with probability p and loses one chip from me with probability $1 - p$.

   (a) If $p = 1/6$, find the smallest number a so that Paul has better than a fifty percent chance of winning all of the chips eventually.

   (b) For this value of a, how long is the game expected to last?

   (c) For this value of a, how many times do you expect Paul's fortune to be a before the game ends?

   Solution: I screwed up. The best Paul can do is 20%. Everyone gets full credit.

2. Consider this problem from genetics. At one particular genetic locus, each animal has a pair of alleles chosen from a and b. The order does not matter, so the three possible genotypes are aa, ab, and bb. When two animals mate, each parent passes on one of its two alleles chosen uniformly at random.

   Consider now this breeding experiment. Initally, we have two animals with genotypes aa and ab. These animals produce two offspring of different sex with their genotypes randomly and independently determined. These siblings are then crossed. This continues indefinitely.

   (a) The state is the unordered pair of genotypes of the parents of the next generation. Find the state space.

   (b) Find the probability transition matrix of this state space. Put it into canonical form so any absorbing states are numbered first. Identify the absorbing states.

   (c) Compute the mean time until absorption.

   (d) Find the probability that the a allele disappears eventually.

   Solution:

   (a) The state space is $S = \{aa/aa, bb/bb, ab/ab, aa/bb, aa/ab, ab/bb\}$. (The order does not matter.)

(b) To find the Markov probability transition matrix, first find the probability of any single offspring for each cross.

parents	aa	ab	bb
aa/aa	1	0	0
bb/bb	0	0	1
ab/ab	1/4	1/2	1/4
aa/bb	0	1	0
aa/ab	1/2	1/2	0
ab/bb	0	1/2	1/2

It remains to find the chance of each pair of offspring. If the offspring are the same, it is just the square of the probability of one. If the offspring are different, it is twice the product of the probabilities of each because they could occur in two different orders. The resulting probability transition matrix with states ordered as above is then

$$\mathbf{P} = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1/16 & 1/16 & 1/4 & 1/8 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 0 & 0 & 1/2 \end{array}\right]$$

(c) Call the lower 4×4 matrix $\mathbf{T}$. The expected number of visits to state j given you begin in state i is $\mathbf{U} = (\mathbf{I} - \mathbf{T})^{-1}$. This matrix is

$$\mathbf{U} = \begin{bmatrix} 2\frac{2}{3} & \frac{1}{3} & 1\frac{1}{3} & 1\frac{1}{3} \\ 2\frac{2}{3} & 1\frac{1}{3} & 1\frac{1}{3} & 1\frac{1}{3} \\ 1\frac{1}{3} & \frac{1}{6} & 2\frac{2}{3} & \frac{2}{3} \\ 1\frac{1}{3} & \frac{1}{6} & \frac{2}{3} & 2\frac{2}{3} \end{bmatrix}$$

The expected absorption time beginning in state i is the corresponding row sum. Beginning in state aa/ab, this is $4\frac{5}{6}$.

(d) The probability the a allele disappears is the probability that the Markov chain is absorbed into state bb/bb. If we let $\mathbf{S}$ be the lower right matrix of $\mathbf{P}$, the absorption probabilities are $\mathbf{A} = \mathbf{US}$. This is

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

The probability of absorption to state bb/bb begining in state aa/ab is 0.25.

3. Consider the following random walk on a tree. Nodes a, b, c, d, e, f, g, and h are connected with the following edge set: $\{(a, f), (b, f), (f, g), (c, g), (g, h), (e, h), (f, h)\}$. At each time, the next node is selected uniformly at random from the neighboring nodes.

   (a) This Markov chain is finite and irreducible. What is the periodicity of the only recurrent class?

   (b) Construct the probability transition matrix P.

   (c) How many eigenvalues have an absolute value strictly less than one?

   (d) What happens to P^n as $n \to \infty$?

   (e) Describe all solutions to the equations $\pi P = \pi$ and $\sum_i \pi_i = 1$. Is there a unique solution or are there many?

   (f) Describe all solutions to the equations $\pi P^2 = \pi$ and $\sum_i \pi_i = 1$. Is there a unique solution or are there many?

   Solution: I messed up the graph, so everyone gets full credit.

4. Redo the previous problem, but consider each node to be a neighbor of itself. For example, from state a you remain at a with probability $1/2$ and move to state f with probability $1/2$.

   Solution: I messed up the graph, so everyone gets full credit.