CPMA 521, Fall 2001
Professor Larget
Assignment #6: due October 15, 2001

I expect that you will use S-PLUS or R for many or all of these problems.

1. Consider a corrected version of the game Paul and I played in class. Paul begins with a chips and I begin with one chip. At each stage, Paul wins one chip from me with probability p and loses one chip from me with probability $1-p$.

   (a) If $p = 2/5$, find the smallest number a so that Paul has better than a fifty percent chance of winning all of the chips eventually.

   (b) For this value of a, how long is the game expected to last?

   (c) For this value of a, how many times do you expect Paul's fortune to be a before the game ends?

   Solution: When $a = 2$, $b = 1$, and $p = 2/5$, the probability that Player A eventually wins is

$$\frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^{a+b} - 1} = \frac{\left(\frac{.6}{.4}\right)^2 - 1}{\left(\frac{.6}{.4}\right)^3 - 1} = 10/19 > 0.5.$$

   The Markov probability transition matrix is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

   The matrix of expected visits is

$$\mathbf{U} = \left(\mathbf{I} - \begin{bmatrix} 0 & 0.4 \\ 0.6 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -0.4 \\ -0.6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 25/19 & 10/19 \\ 15/19 & 25/19 \end{bmatrix} \doteq \begin{bmatrix} 1.32 & 0.53 \\ 0.79 & 1.32 \end{bmatrix}$$

   The game is expected to last $0.79 + 1.32 = 2.11$ turns on average.

   You expect that Paul's fortune will be two chips 1.32 times on average.

2. Consider the following random walk on a tree. Nodes a, b, c, d, e, f, g, and h are connected with the following edge set: $\{(a,f),(b,f),(f,g),(c,g),(g,h),(d,h),(e,h)\}$. At each time, the next node is selected uniformly at random from the neighboring nodes.

   (a) This Markov chain is finite and irreducible. What is the periodicity of the only recurrent class?

   (b) Construct the probability transition matrix P.

(c) How many eigenvalues have an absolute value strictly less than one?

(d) What happens to P^n as $n \to \infty$?

(e) Describe all solutions to the equations $\pi P = \pi$ and $\sum_i \pi_i = 1$. Is there a unique solution or are there many?

(f) Describe all solutions to the equations $\pi P^2 = \pi$ and $\sum_i \pi_i = 1$. Is there a unique solution or are there many?

Solution:

(a) The periodicity is 2.

(b)

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 \end{bmatrix}$$

(c) Six of the eight eigenvalues are strictly less than 1. One is 1 and one is -1.

(d) As $n \to \infty$, $\mathbf{P}^n$ alternates between two limit matrices, one for odd n and one for even. For odd n, $\mathbf{P}^n$ tends to

$$\begin{bmatrix} 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \end{bmatrix}$$

while for even n, it tends to

$$\begin{bmatrix} 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \\ 1/7 & 1/7 & 0 & 1/7 & 1/7 & 0 & 3/7 & 0 \\ 0 & 0 & 1/7 & 0 & 0 & 3/7 & 0 & 3/7 \end{bmatrix}$$

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(e) The unique solution to $\pi\mathbf{P} = \pi$ subject to the constraint that $\sum \pi_i = 1$ is

$$\pi = (1/14, 1/14, 1/14, 1/14, 1/14, 3/14, 3/14, 3/14).$$

(f) Solutions to $\pi\mathbf{P}^2 = \pi$ subject to $\sum \pi_i = 1$ and $\pi \geq 0$ are not unique but are of the form

$$\pi = \alpha(1/7, 1/7, 0, 1/7, 1/7, 0, 3/7, 0) + (1-\alpha)(0, 0, 1/7, 0, 0, 3/7, 0, 3/7)$$

where $0 \leq \alpha \leq 1$. The reason is that the chain is of periodicity 2. If we take two consecutive steps each time, we will always stay in the same periodic class. Any linear combination of the two separate stationary distributions for the classes will be a solution.

3. Redo the previous problem, but consider each node to be a neighbor of itself. For example, from state a you remain at a with probability $1/2$ and move to state f with probability $1/2$.

Solution:

(a) The periodicity is 1.

(b)

$$\mathbf{P} = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 \end{bmatrix}$$

(c) Seven of the eight eigenvalues are strictly less than 1. One is 1.

(d) As $n \to \infty$, $\mathbf{P}^n$ tends to

$$\begin{bmatrix} 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \\ 1/11 & 1/11 & 1/11 & 1/11 & 1/11 & 2/11 & 2/11 & 2/11 \end{bmatrix}$$

(e) The unique solution to $\pi\mathbf{P} = \pi$ subject to the constraint that $\sum \pi_i = 1$ is

$$\pi = (1/11, 1/11, 1/11, 1/11, 1/11, 2/11, 2/12, 2/12).$$

(f) Solutions to $\pi \mathbf{P}^2 = \pi$ subject to $\sum \pi_i = 1$ and $\pi \geq 0$ are also unique and are the same as the previous part.

4. In class we discussed three descriptions of a Poisson process. I want you to empirically verufy that two of the descriptions are equivalent.

   (a) Write an S program to place a Poisson number of points on a line segment, with their locations chosen uniformly at random. The function should allow the user to specify the segment length and the rate of the Poisson process. User this function to simulate a Poisson process with rate 1 on a segment of length 100. Then, compute the interarrival times (try something like `diff(sort(c(0,x)))`). Make a quantile-quantile plot of these random points versus a random sample of the same size of random exponential random variables with rate 1 generated using `rexp`. See Krause and Olson page 182 for a description of `qqplot`. If most of the points fall close to a line, this is evidence they come from the same distribution.

   (b) Write an S program to generate a sequence of exponential random variables as interarrival times of a point process. See how many events fall into the interval (0,2). Repeat this thousands of times. How do the long-run relative frequencies compare to a Poisson(2) distribution probabilities? You will use the functions `rexp` to generate exponential random variables, `cumsum` to find the locations of the point, and `dpois` to calculate Poisson probabilities.

Solution: One hundred is too small to make the plot straight with high probability. But 10,000 works pretty well. Here is a sample function.

```
prob4a <- function(lambda=1,tt=100) {
  n <- rpois(1,lambda*tt)
  x <- runif(n)*tt
  e <- diff(c(0,sort(x)))
  qqplot(e,rexp(n))
  invisible()
}
```

For part (b), here is a function that works.

```
prob4b <- function(lambda=1,tt=2,ntrials=1) {
  counts <- rep(0,10*tt/lambda) # create an array to store the counts
  for(i in 1:ntrials) {
    e <- rexp(10*tt/lambda)
    while(sum(e)<tt)
      e <- c(e,rexp(10*tt/lambda))
    n <- sum(cumsum(e)<tt)
    counts[n+1] <- counts[n+1]+1
  }
```

```
  counts
}
```

In one realization with `ntrials=10000`, here are the counts compared to the expected counts from a Poisson(2) distribution. Notice the similarities.

k	0	1	2	3	4	5	6	7	8	9
observed	1310	2694	2694	1816	961	350	134	29	10	2
expected	1353	2707	2707	1804	902	361	120	34	9	2