Fact Sheet for Final Exam

General solutions of the system of linear differential equations $\dot{X} = CX$ when C is a 2×2 matrix.

case	eigenvalues	eigenvectors	solution
Real distinct eigenvalues	λ_1 and λ_2	$\boldsymbol{v_1}$ and $\boldsymbol{v_2}$	$X(t) = \alpha_1 e^{\lambda_1 t}\boldsymbol{v}_1 + \alpha_2 e^{\lambda_2 t}\boldsymbol{v}_2$
Complex conjugate eigenvalues	$\sigma \pm i\tau$	$\boldsymbol{v} \pm i\boldsymbol{w}$	$X(t) = \alpha_1 e^{\sigma t}(\cos(\tau t)\boldsymbol{v} - \sin(\tau t)\boldsymbol{w}) + \alpha_2 e^{\sigma t}(\sin(\tau t)\boldsymbol{v} + \cos(\tau t)\boldsymbol{w})$
Equal eigenvalues, one eigenvector, one generalized eigenvector	λ	$\boldsymbol{v}$, $(\boldsymbol{w})$	$X(t) = e^{\lambda t}(\alpha_1 \boldsymbol{v} + \alpha_2(\boldsymbol{w} + t\boldsymbol{v}))$
Equal eigenvalues, two eigenvectors	λ	$\boldsymbol{v_1}$, $\boldsymbol{v_2}$	$X(t) = \alpha_1 e^{\lambda t}\boldsymbol{v}_1 + \alpha_2 e^{\lambda t}\boldsymbol{v}_2$

Definition of generalized eigenvector: If C has exactly one linearly independent real eigenvector $\boldsymbol{v}$ with real eigenvalue λ, $\boldsymbol{w}$ is a generalized eigenvector of C if $(C - \lambda I)\boldsymbol{w} = \boldsymbol{v}$.

Solutions to normal forms of linear planar systems:

Equation	Closed form solution
$\dot{X} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} X$	$X(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} X_0$
$\dot{X} = \begin{pmatrix} \sigma & -\tau \\ \tau & \sigma \end{pmatrix} X$	$X(t) = e^{\sigma t}\begin{pmatrix} \cos(\tau t) & -\sin(\tau t) \\ \sin(\tau t) & \cos(\tau t) \end{pmatrix} X_0$
$\dot{X} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} X$	$X(t) = e^{\lambda_1 t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} X_0$

Cookbook for first order ODEs:

Separable: $dx/dt = f(t)g(x)$

Linear: $dx/dt + a(t)x = g(t)$
General solution $x(t) = c(t)x_h(t)$ where $x_h(t)$ is the solution to the homogeneous equation and $c'(t) = g(t)/x_h(t)$.

Homogeneous Coefficients: $dx/dt = F(x/t)$
The substitution $v = x/t$ transforms the equation to $dv/dt = (F(v) - v)/t$ which is separable.

Bernoulli: $dx/dt + a(t)x = g(t)x^p$
The substitution $v = x^{1-p}$ leads to the equation $dv/dt + (1-p)a(t)v = (1-p)g(t)$ which may be solved by another method.

Exact: $h(t,x)dx + (-g(t,x))dt = 0$
The equation is exact if $-g_x(t,x) = h_t(t,x)$. The solution satisfies $F(t,x) = C$ where $F_x(t,x) = h(t,x)$ and $F_t(t,x) = -g(t,x)$.