Problem Set 1 - Solution to problem 4

Step 1: *Evaluate the f(n) for n=1,2,3,4,... until we see a pattern.*

$$
\begin{aligned}
& & f(1) &= 2001 \quad (given) \\
f(1)+f(2) = 2^2 f(2) & \implies & f(2) &= 2001 \cdot \frac{1}{2^2-1} \\
& & &= 2001 \cdot \frac{1}{3 \cdot 1} = 2001 \cdot \frac{2}{3 \cdot 2} \\
f(1)+f(2)+f(3) = 3^2 f(3) & \implies & f(3) &= 2001 \cdot \frac{2^2}{3^2-1} \cdot \frac{1}{2^2-1} \\
& & &= 2001 \cdot \frac{2 \cdot 2}{4 \cdot 2} \frac{1}{3 \cdot 1} = 2001 \cdot \frac{2}{4 \cdot 3} \\
f(1)+f(2)+f(3)+f(4) = 4^2 f(4) & \implies & f(4) &= 2001 \cdot \frac{3^2}{4^2-1} \cdot \frac{2^2}{3^2-1} \cdot \frac{1}{2^2-1} \\
& & &= 2001 \cdot \frac{3 \cdot 3}{5 \cdot 3} \cdot \frac{2 \cdot 2}{4 \cdot 2} \cdot \frac{1}{3 \cdot 1} = 2001 \frac{2}{5 \cdot 4}
\end{aligned}
$$

Step 2: *Look for a pattern and try to come up with a general formula for f(n).*

$$
\begin{aligned}
f(n) &= 2001 \cdot \frac{(n-1)^2}{n^2-1} \cdot \frac{(n-2)^2}{(n-1)^2-1} \cdot \frac{(n-3)^2}{(n-2)^2-1} \cdots \frac{3^2}{4^2-1} \cdot \frac{2^2}{3^2-1} \cdot \frac{1}{2^2-1} \\
&= 2001 \cdot \frac{(n-1)(n-1)}{(n+1)(n-1)} \cdot \frac{(n-2)(n-2)}{n(n-2)} \cdot \frac{(n-3)(n-3)}{(n-1)(n-3)} \cdot \cdots \frac{3 \cdot 3}{5 \cdot 3} \cdot \frac{2 \cdot 2}{4 \cdot 2} \cdot \frac{1}{3 \cdot 1} \text{ (after factoring)} \\
&= 2001 \cdot \frac{2}{n(n+1)} \text{ (after canceling)}
\end{aligned}
$$

Step 3: *Evaluate f(2001).*

$$
\begin{aligned}
f(2001) &= 2001 \cdot \frac{2}{2001 \cdot 2002} \\
&= \frac{1}{1001}
\end{aligned}
$$