Solution to Problem # 2, Problem Set #2

2–2: The sequence of integers $u_0, u_1, u_2, u_3, ...$ satisfies $u_0 = 1$ and $u_{n+1}u_{n-1} = ku_n$ for each $n \geq 1$, where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

SOLUTION: Let's first solve the expression for u_{n+1}, so that $u_{n+1} = ku_n/u_{n-1}$ for $n \geq 1$. Then, let's substitute in for the numerator.

$$u_{n+1} = \frac{ku_n}{u_{n-1}} = \frac{k \cdot ku_{n-1}/u_{n-2}}{u_{n-1}} = \frac{k^2}{u_{n-2}}$$

We see a pattern for every third value. We can write it more clearly by adding two to each subscript, so $u_{n+3} = k^2/u_n$. Notice then what happens if we apply this new result to u_n.

$$u_{n+3} = \frac{k^2}{u_n} = \frac{k^2}{k^2/u_{n-3}} = u_{n-3}$$

Again, by renumbering indices, we state clearly that the u_i must form a periodic sequence of length six, $u_{n+6} = u_n$.

Therefore, because $2000 \mod 6 = 2$, $2000 = u_{2000} = u_2$. We can write out the entire periodic sequence, also using the fact that each u_i is the product of its two neighbor values over k. This implies that $u_1 = u_0u_2/k = 2000/k$, for example.

$$u_0 = 1, \qquad u_1 = \frac{2000}{k}, \qquad u_2 = 2000, \qquad u_3 = k^2, \qquad u_4 = \frac{k^3}{2000}, \qquad u_5 = \frac{k^2}{2000}, \qquad u_6 = 1$$

Now we are also given that each u_i is an integer. This means that k must divide evenly into 2000 and that 2000 must divide evenly into k^2 and k^3. We can factor $2000 = 2^4 5^3$, so k must be of the form $k = 2^x 5^y$. Thus,

$$\frac{2000}{k} = \frac{2^4 5^3}{2^x 5^y}$$

which implies that $x \leq 4$ and $y \leq 3$. Also,

$$\frac{k^2}{2000} = \frac{2^{2x} 5^{2y}}{2^4 5^3}$$

so $4 \leq 2x$, and $3 \leq 2y$. Putting this all together, $x \in \{2, 3, 4\}$ and $y \in \{2, 3\}$. Each of the six combinations is a solution.

			Pattern						
x	y	k	u_0	u_1	u_2	u_3	u_4	u_5	u_6
2	2	100	1	20	2000	10,000	500	5	1
3	2	200	1	10	2000	40,000	4000	20	1
4	2	400	1	5	2000	160,000	32,000	80	1
2	3	500	1	4	2000	250,000	62,500	125	1
3	3	1000	1	2	2000	1,000,000	500,000	500	1
4	3	2000	1	1	2000	4,000,000	4,000,000	2000	1

Solution by Bret Larget.