Math 340–Problem Solving Seminar, Fall 2001, Problem Set 2

(1) Starting with any three-digit number n (such as $n = 625$), we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs, and the product of all three digits.

(a) Find the value of $\frac{n}{f(n)}$ when $n = 625$. (The answer is an integer!)

(b) Find all three-digit numbers n such that the ratio $\frac{n}{f(n)} = 1$.

(2) The sequence of integers $u_0, u_1, u_2, u_3, \ldots$ satisfies $u_0 = 1$ and

$$u_{n+1}u_{n-1} = ku_n \text{ for each } n \geq 1,$$

where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

(3) Let a and b be positive integers. Prove that

$$4(a^3 + b^3) \geq (a + b)^3$$

(4) Given any real number $a \neq -1$, the sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = a \text{ and } x_{n+1} = x_n^2 + x_n \text{ for all } n \geq 1.$$

Let

$$y_n = \frac{1}{(1 + x_n)}, \quad S_n = \sum_{i=1}^{n} y_i, \quad P_n = \prod_{i=1}^{n} y_i.$$

Prove that

$$aS_n + P_n = 1 \text{ for all } n \geq 1.$$

(HINT: First show that $P_n = \frac{a}{x_{n+1}}$.)

(5) Given that x is a positive integer, find all solutions of

$$\left[\sqrt[3]{1}\right] + \left[\sqrt[3]{2}\right] + \cdots + \left[\sqrt[3]{(x^3 - 1)}\right] = 400.$$

Note: $[z]$ denotes the largest integer $\leq z$.