Math 340 — Problem Solving Seminar, Fall 2001, Problem Set #3

Please give each of these problems a try. We do not expect you to solve them all.

Problem 1: The seven dwarves. As the seven dwarves head off to work each day, they sing their favorite song, "High low, high low, it's off to work we go." If the seven dwarves have seven distinct heights, in how many ways can they line up in single file so that the first, third, fifth, and seventh dwarves are each taller than their neighbors in line?

Problem 2: A coin-tossing game. A biased coin with head probability p is tossed repeatedly and independently until either the pattern "HHH" or "HTH" appears. For what value of p are these two patterns equally likely to appear first?

Problem 3: High octagon. An octagon is inscribed in a circle. One set of four nonconsecutive vertices of the octagon are the vertices of a square with area 5. The other set of four nonconsecutive vertices of the octagon are the vertices of a rectangle with area 4. Find the maximum possible area of the octagon.

Problem 4: An odd problem. Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume for each j that at least one of a_j, b_j, and c_j is odd. Show that there exist integers r, s, and t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j, 1 \leq j \leq N$.