Math 340–Problem Solving Seminar, Fall 2001, Problem Set 6

(1) A non-negative integer $f(n)$ is assigned to each positive integer n in such a way that the following conditions are satisfied:

(a) $f(mn) = f(m) + f(n)$, for all positive integers m, n;

(b) $f(n) = 0$ whenever the units digit of n (in base 10) is a '3'
(i.e., $f(3) = 0$, $f(13) = 0$, $f(23) = 0$, ...); and

(c) $f(10) = 0$.

Prove that $f(n) = 0$, for all positive integers n.

(2) Given any three numbers a, b, and c between 0 and 1, prove that not all of the expressions $a(1-b)$, $b(1-c)$, and $c(1-a)$ can be greater than $\frac{1}{4}$.

(3) Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

(4) Given a point (a, b) with $0 > b > a$, determine the minimum perimeter of a triangle with one vertex at (a, b) one on the x-axis, and one on the line $y = x$.