Math 340 — Problem Solving Seminar, Fall 2001, Problem Set #7

Problem 1: Prime sets. The first n prime numbers 2, 3, 5, ..., p_n are partitioned into two sets, $A = \{a_1, a_2, \ldots, a_h\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ with $h + k = n$. The numbers $\{\alpha_i\},\ i = 1, \ldots, h$ and $\{\beta_j\},\ j = 1, \ldots, k$ are positive integers. Let

$$x = \prod_{i=1}^{h} a_i^{\alpha_i} \qquad \text{and} \qquad y = \prod_{j=1}^{k} b_j^{\beta_j}.$$

If positive integer d divides evenly into $x - y$, prove that either $d = 1$ or $d > p_n$. [BMO 11–2]

Problem 2: Uniform points. A stick of length one is broken at n points where each break point is chosen independently and uniformly at random. Find as a function of n the probability that the longest segment of the stick has length longer than one half. *(Hint: First find the probability that the first segment is the longest and its length is greater than one half.)* [BRL 2001]

Problem 3: Convex quadrilateral. A convex quadrilatral has angles A, B, C, and D and satisfies

$$\cos A + \cos B + \cos C + \cos D = 0$$

Prove that the quadrilateral must either be cyclic (meaning all four vertices lie on a circle) or trapezoidal (meaning that at least one set of opposite sides are parallel).

Problem 4: 2×2 integer matrices. Suppose that A and B are 2×2 matrices with integer entries. Furthermore, A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are each invertible and the inverses have integer entries. Show that $A + 5B$ is invertible and the inverse has integer entries as well. [P–1994]