Math 340–Problem Solving Seminar, Fall 2001, Problem Set 8

(1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube? (P98A1)

(2) In a party of ten people, you are told that among any three people there are at least two who do not know each other. Prove that the party contains at least four people, none of whom know the other three. (BMO16-5)

(3) Given positive integers m and n, S_m is equal to the sum of m terms of the series

$$(n + 1) - (n + 1)(n + 3) + (n + 1)(n + 2)(n + 4)$$
$$- (n + 1)(n + 2)(n + 3)(n + 5) + \cdots$$

the terms of which alternate in sign, with each term (after the first) equal to the product of consecutive integers with the last but one integer being omitted. Prove that S_m is divisible by $m!$ but not necessarily by $m!(n + 1)$. (BMO17-2)

(4) Let $P(x)$ be the polynomial

$$P(x) = (x - \cos 36^o)(x - \cos 84^o)(x - \cos 156^o).$$

Prove that the coefficients of x^2 and x are rational while that of the constant term is irrational. (MH S-54)