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3D Eigenfunction Expansion of Sparsely Sampled 2D Cortical Data

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Introduction

We present a new explicit functional representation technique to address the problem of resampling sparsely sampled 2D cortical data to a densely defined 3D volume space. The cortical data is represented as the linear combination of the eigenfunctions of the 3D Laplacian. Our approach should offer more unified modeling flexibility than widely used radial basis approaches [1].

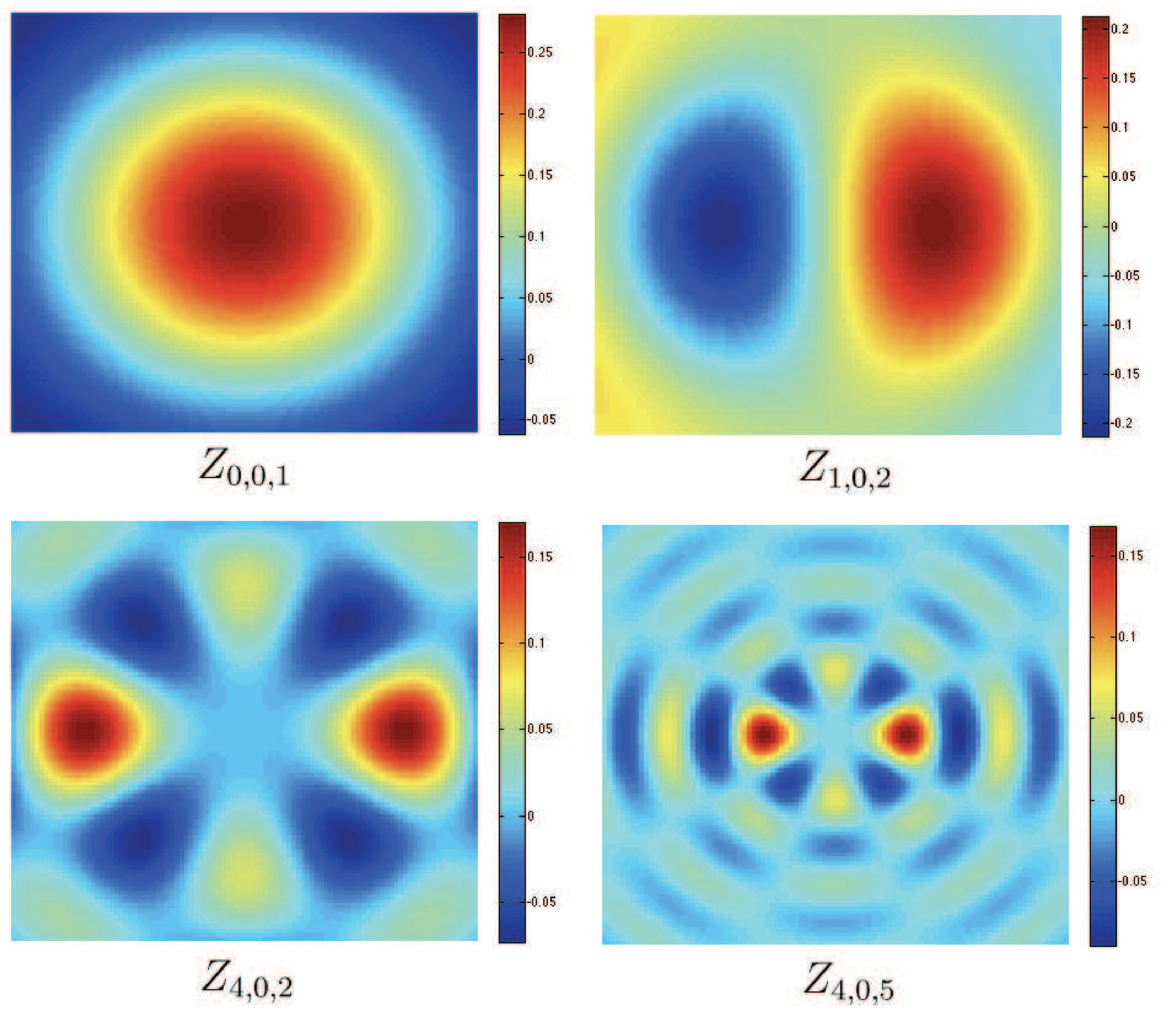

Figure 1: Basis functions Z_{lmn} are visualized in the cube $[-1,1]^3$ at the cross section $p_2 = 0$. The expansion is only valid within the ball of radius 1. Z_{lmn} are the multiples of spherical harmonics and spherical Bessel functions

Eigenfunction Expansion

Suppose the Cartesian coordinates (p_1, p_2, p_3) are given by the spherical coordinates (r, θ, φ) as

$$(p_1, p_2, p_3) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta).$$

Consider the eigenvalue problem

$$\Delta f + \lambda f = 0$$

in the solid ball of radius 1 with the Dirichlet boundary condition

$$f(r = 1, \theta, \varphi) = 0.$$

The eigenfunctions of the 3D Laplacian are given by

$$Z_{lmn}(r, \theta, \varphi) = S_l(\sqrt{\lambda_{ln}} r) Y_{lm}(\theta, \varphi),$$

where Y_{lm} are the spherical harmonics of degree l and order m [2], and S_l are the spherical Bessel function (Figure 1). The roots of the spherical Bessel function, i.e. $S_l(x) = 0$, are ordered as

$$0 < \sqrt{\lambda_{l,1}} < \sqrt{\lambda_{l,2}} < \sqrt{\lambda_{l,3}} < \cdots.$$

Then any square integrable function can be expanded as

$$f(r, \theta, \varphi) \approx \sum_{l=0}^{k} \sum_{m=-l}^{l} \sum_{n=1}^{j} \beta_{lmn} Z_{lmn}(r, \theta, \varphi),$$

where the degree k and the number of roots j have to be given a priori.

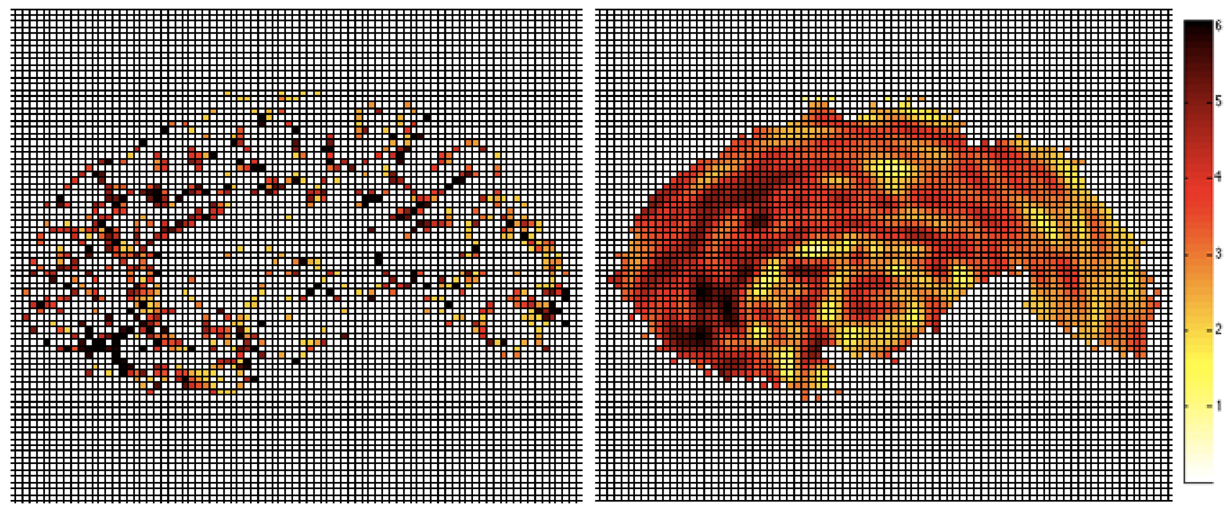

Figure 2: Left: 2D cortical thickness defined on mesh vertices are resampled on 3D voxel space. Right: Eigenfunction expansion with degree k= 22 and j=22 roots. Only the masked brain region is shown. The representation can fill out voxels smoothly where cortical thickness is not defined.

Iterative Residual Fitting Algorithm

Previously the coefficients of spherical harmonic series expansion have been estimated using the least squares method by solving the system of linear equations [2] [3]. For a cortical surface mesh with N vertices, where N can reach upwards of 400000 for cortical meshes, we need to simultaneously solve for N equations and, in turn, invert an $N \times N$ matrix. To address this computational bottleneck, we have developed the *iterative residual fitting* algorithm [2] that divide s the extremely large linear problem into smaller subset of linear problems.

Let $p_i = (r_i, \theta_i, \varphi_i)$ be the mesh vertices. We vectorize the measurement as

$$\mathbf{f} = (f(p_1), \cdots, f(p_N))'.$$

Let $\mathbf{Z}_{l,\cdot,n}$ be the $N \times (2l+1)$ submatrix of basis:

$$\mathbf{Z}_{l,\cdot,n} = \begin{bmatrix} Z_{l,-l,n}(p_1) & \cdots & Z_{l,l,n}(p_1) \\ \vdots & \ddots & \vdots \\ Z_{l,-l,n}(p_N) & \cdots & Z_{l,l,n}(p_N) \end{bmatrix}.$$

Denote the matrix of all basis corresponding to the l-th degree as $\mathbf{Z}_l = [\mathbf{Z}_{l,\cdot,1}, \cdots, \mathbf{Z}_{l,\cdot,j}]$. Define the vector of coefficients corresponding to $\mathbf{Z}_l$ as

$$\beta_l = (\beta_{l,-l,1}, \cdots, \beta_{l,l,j})'.$$

Then we iteratively estimate the coefficients of low degrees to high degrees using the iterative algorithm:

1. $l \leftarrow 0$.
2. $\mathbf{r} \leftarrow \mathbf{f}$.
3. $\beta_0 \leftarrow (\mathbf{Z}_0 \mathbf{Z}_0)^{-1} \mathbf{Z}_0' \mathbf{f}$.
4. $l \leftarrow l + 1$.
5. $\mathbf{r} \leftarrow \mathbf{r} - \mathbf{Z}_{l-1} \beta_{l-1}$.
6. $\beta_l \leftarrow (\mathbf{Z}_l \mathbf{Z}_l)^{-1} \mathbf{Z}_l' \mathbf{r}$.
7. If $l \le k$, go to Step 4.

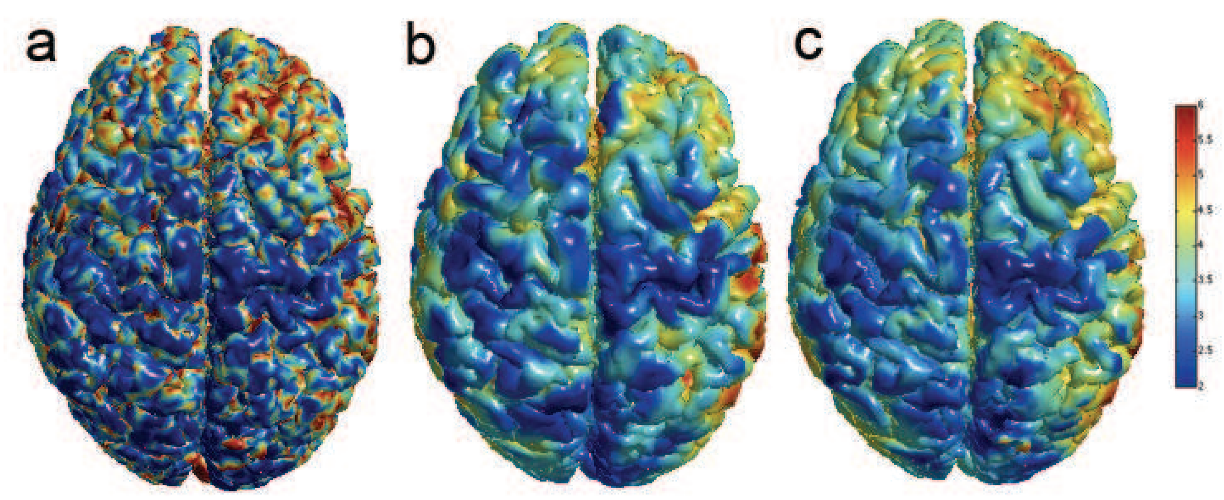

Figure 3: (a) Cortical thickness usually ranges from 2 to 6 mm. (b) Eigenfunction expansion with degree $k = 22$ and $j = 5$ number of roots. (c) Eigenfunction expansion with $k = 10$ and $j = 22$.

3D Resampling of 2D Cortical Data

High resolution magnetic resonance images were obtained using a 3-Tesla GE SIGNA scanner and went through a sequence of image processing steps to obtain cortical thickness defined on mesh vertices [4] (Figure 2 and 3). We have scaled the mesh coordinates to be contained in the ball of radius 1. Then we have performed the eigenfunction expansion. As the degree and the number of roots increases, the expansion should be able to represent more detailed cortical pattern. Taking the given cortical thickness as the ground truth, we have computed the average relative error (Figure 4).

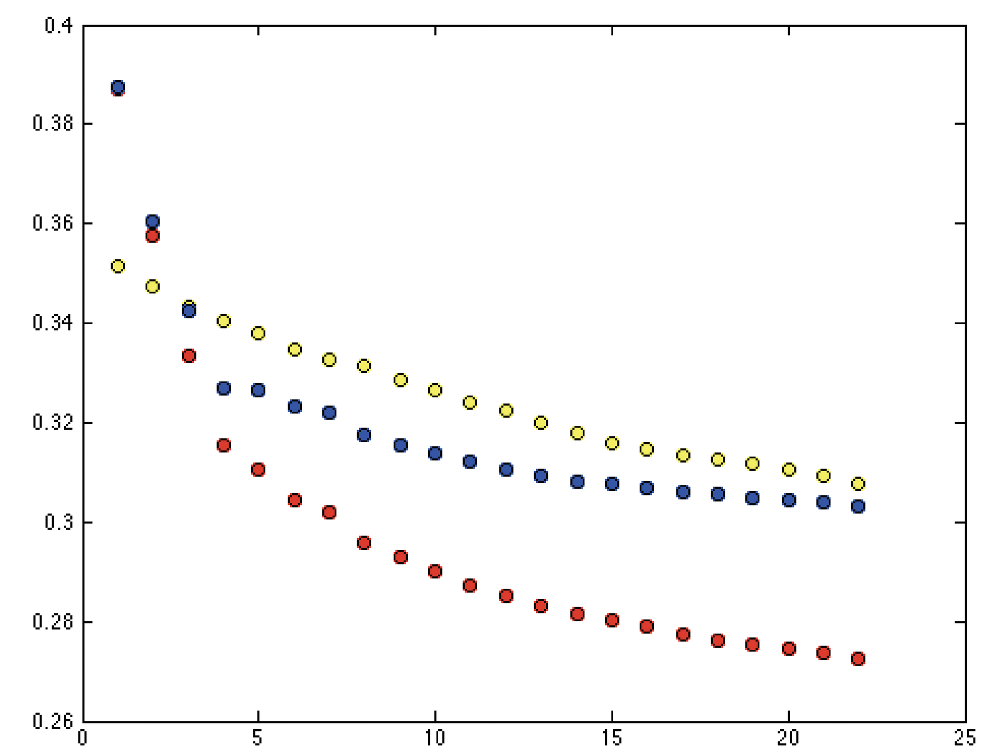

Figure 4: Relative error plot of eigenfunction expansion for various number of degrees and roots. Yellow dots are the errors for varying number of degrees for the fixed number of roots $j = 5$. Red (blue) dots are the errors for varying number of roots at the fixed degree $k = 20$ (10). The plots show that increasing the number of degrees and roots also increases the accuracy.

References

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