

Medical Image Analysis

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Lecture 06.

Spherical Harmonic (SPHARM)
Representation

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Spherical Harmonic (SPHARM) Representation

- Spherical harmonics are basis functions on a unit sphere.
- SPHARM can be used to construct the Fourier series representation of a functional measurement
- Recent development- Wavelet approach, Weighted-SPAHRM
- SPAHRM has been used in parameterizing anatomical boundary

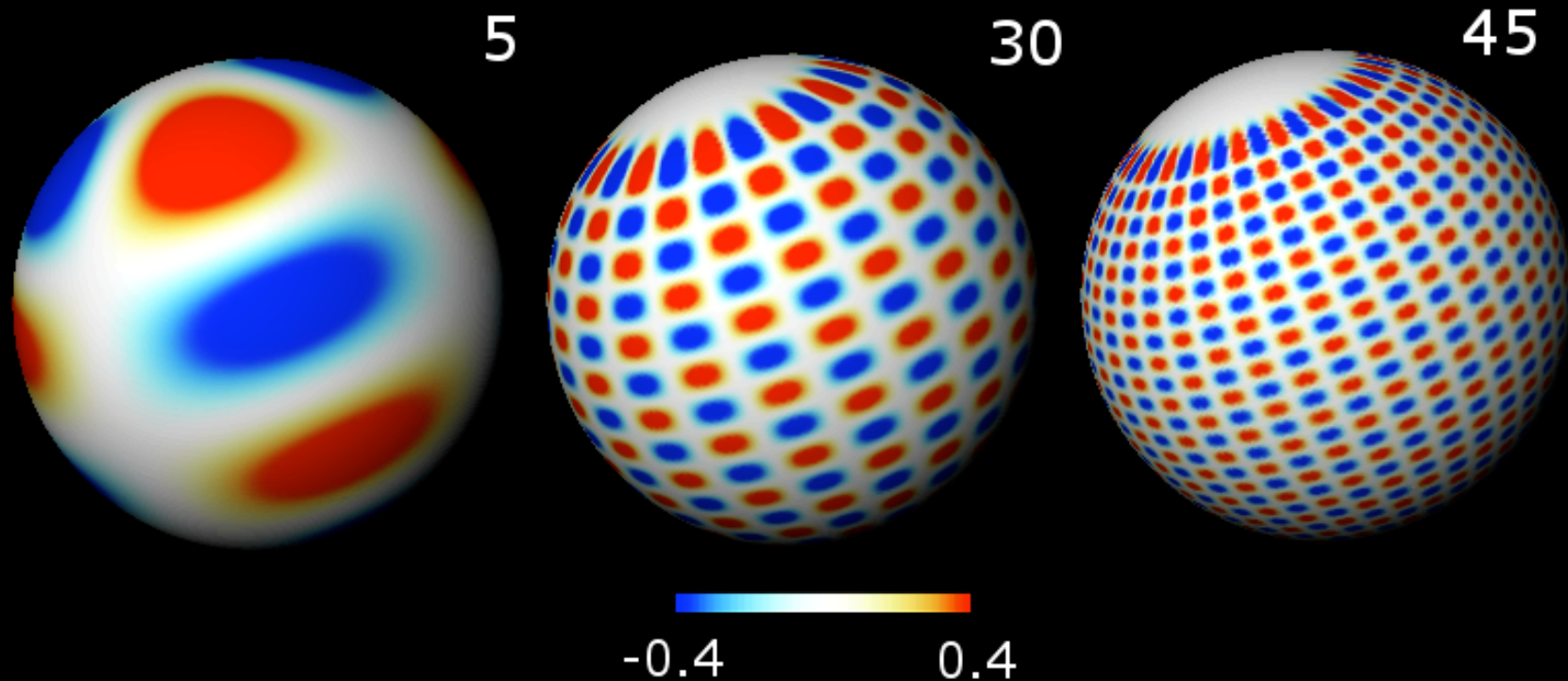
Spherical harmonics

Y_{lm} is called the *spherical harmonic* of degree l and order m .

$$Y_{lm} = \begin{cases} c_{lm} P_l^{|m|}(\cos \theta) \sin(|m|\varphi), & -l \leq m \leq -1, \\ \frac{c_{lm}}{\sqrt{2}} P_l^0(\cos \theta), & m = 0, \\ c_{lm} P_l^{|m|}(\cos \theta) \cos(|m|\varphi), & 1 \leq m \leq l, \end{cases}$$

where $c_{lm} = \sqrt{\frac{2l+1}{2\pi} \frac{(l-|m|)!}{(l+|m|)!}}$ and P_l^m is the associated Legendre polynomials of order m .

Spherical harmonic basis of degree 5, 30 and 45



Lower degree $\leftrightarrow$ Coarse anatomical detail
Higher degree $\leftrightarrow$ Fine anatomical detail

SPHRM representation

- Given functional measurement $f(p)$, it is modeled as

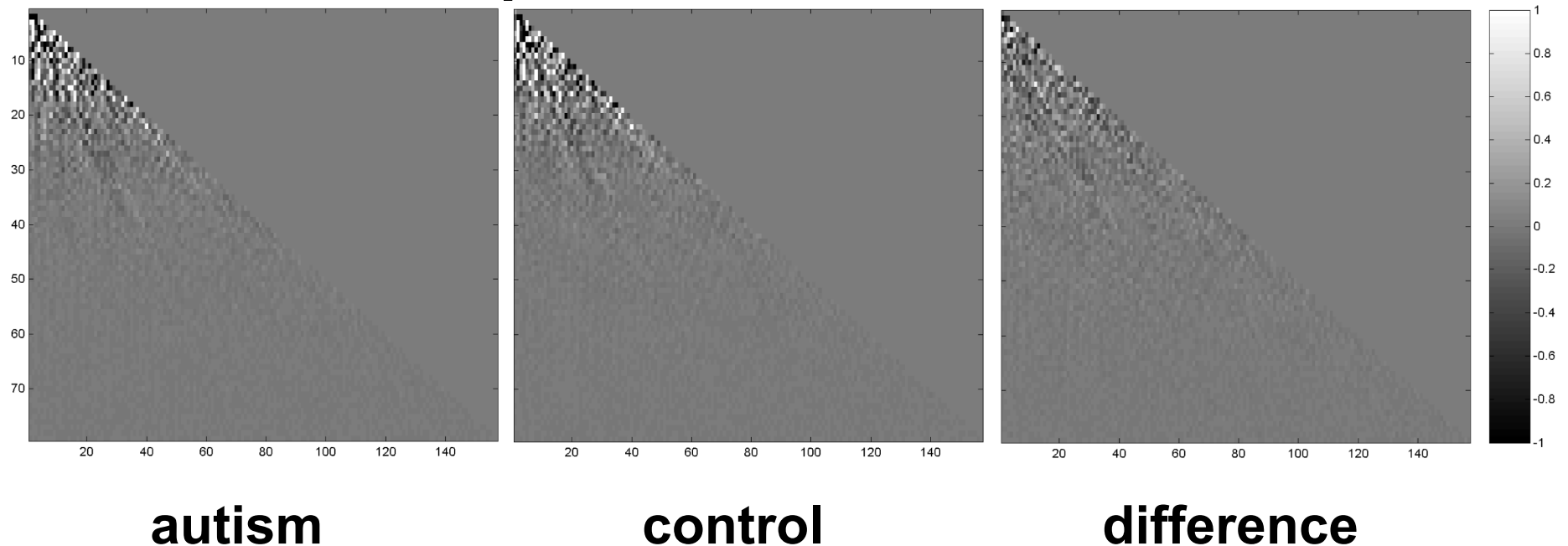
$$f(p) = \sum_{l=0}^k \sum_{m=-l}^l f_{lm} Y_{lm}(p) + e(p)$$

e : noise (image processing, numerical, biological)

f_{lm} : unknown Fourier coefficients

- The parameters are estimated in the least squares fashion.

78th degree SPHARM representation



The coefficients are treated as a multivariate measure and feed into classification techniques.

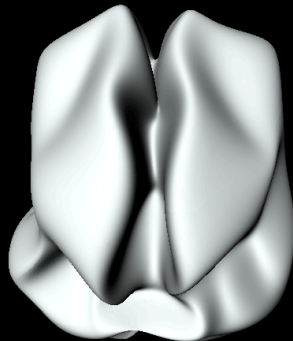
SPHARM representation up to degree 35



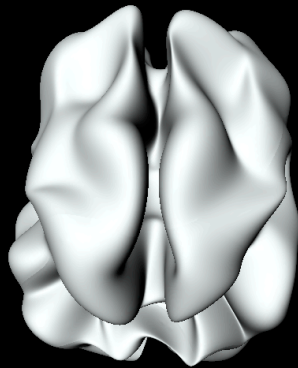
Original cortex



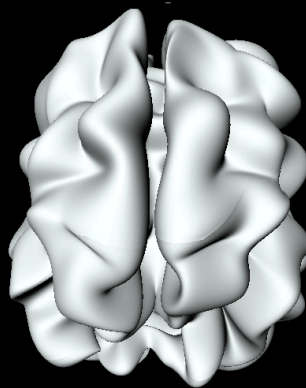
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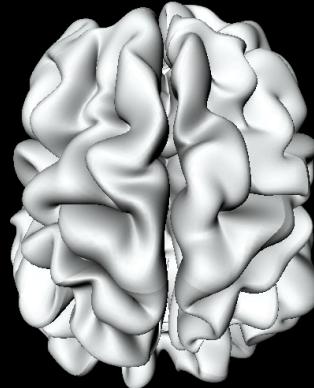
10



15



20



35

- Based on direct numerical integration.
- More than 24 hours of computation.
- We will show you 10min super fast algorithm

Weighted-SPHARM representation

Initial data: i-th Cartesian coordinate


$$\frac{\partial g}{\partial \sigma} = \Delta g, \quad g(p, \sigma = 0) = f(p)$$

Parameter σ controls the amount of smoothing.

The solution is written as the weighted linear combination of spherical harmonics.

Weighted SPHARM is directly related to the following smoothing techniques

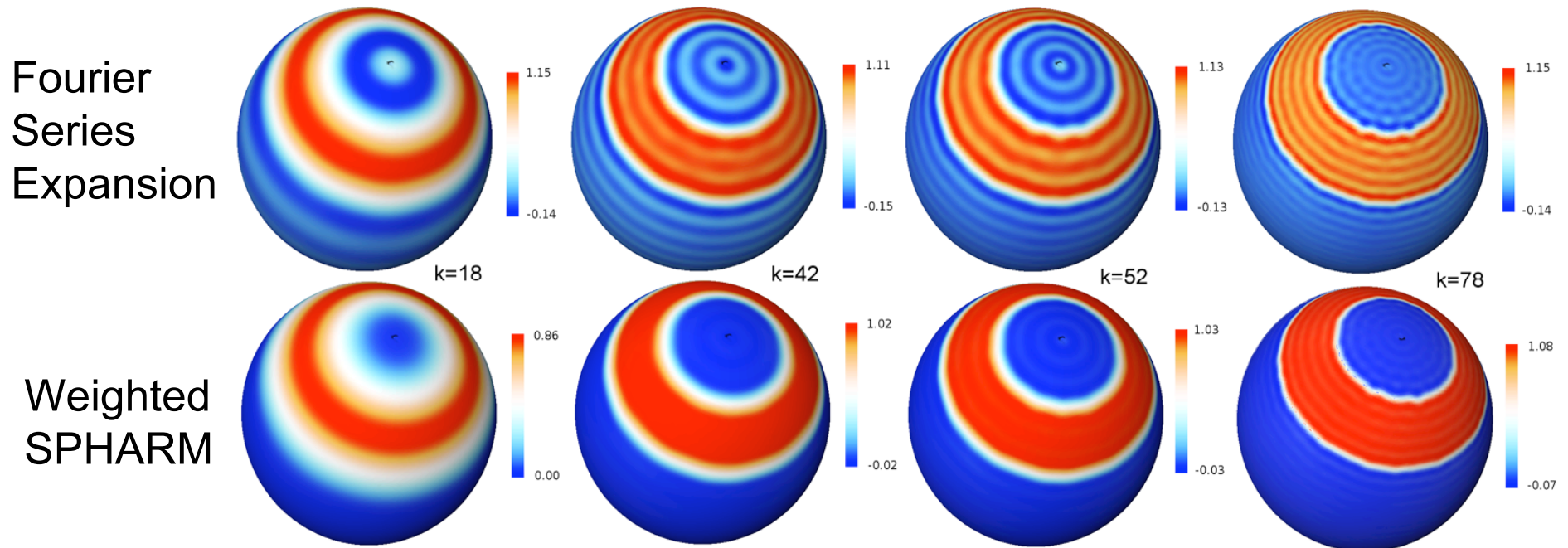
1. Diffusion smoothing

Partial Differential Equation (PDE) based approach with finite element method (FEM).

2. Heat kernel smoothing

Spatially adaptive iterative application of Nadaya-Waton type kernel smoother.

One main advantage of weighted-SPHARM over
Fourier series approach:
Reduction of Gibbs effect (ringing artifacts)



$$\sum_{l=0}^k \sum_{m=-l}^l e^{-l(l+1)\sigma} f_{lm} Y_{lm}$$

Weighted-SPHARM representation

Surface coordinates

$$v(\theta, \varphi) = (v_1(\theta, \varphi), v_2(\theta, \varphi), v_2(\theta, \varphi))$$

Weighted-SPHARM

$$v_i(\theta, \varphi) = \sum_{l=0}^k \sum_{m=-l}^l e^{-l(l+1)\sigma} f_{lm}^i Y_{lm}(\theta, \varphi) + \text{noise}$$

This generalizes the traditional SPHARM representation.

Statistical model on weighted-SPHARM

$$v_i(\theta, \varphi) = \sum_{l=0}^k \sum_{m=-l}^l e^{-l(l+1)\sigma} f_{lm}^i Y_{lm}(\theta, \varphi)$$

$$f_{lm}^i \sim N(\mu_{lm}^i, \sigma_l^2)$$

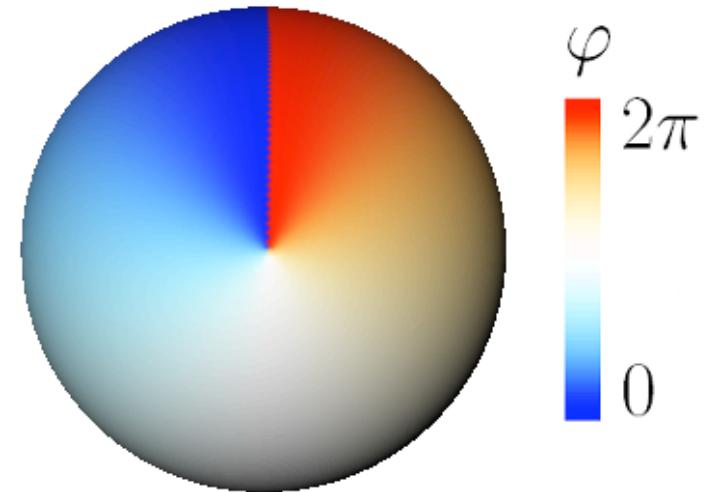
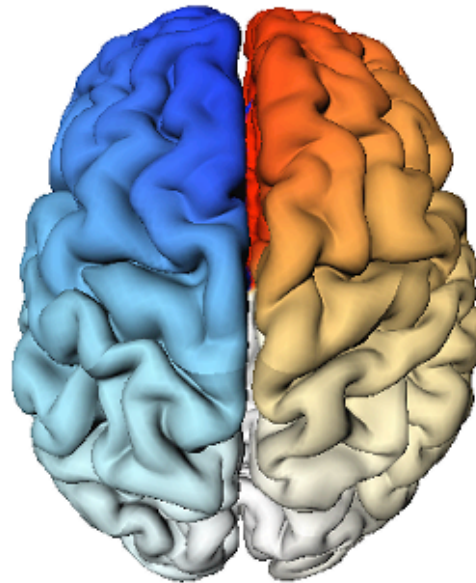


Equivalent to Random Field Theory

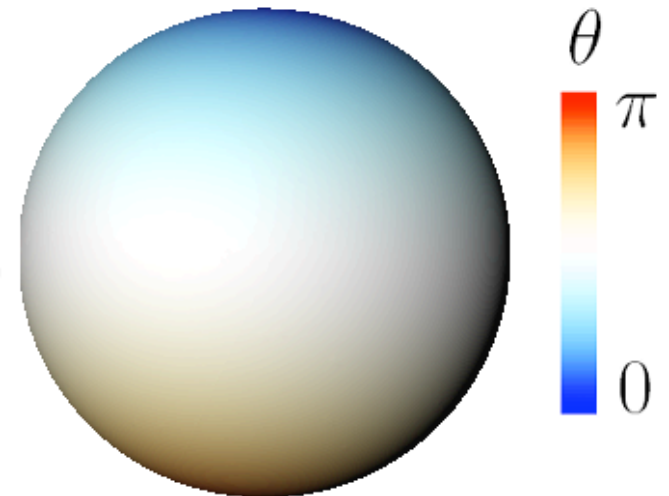
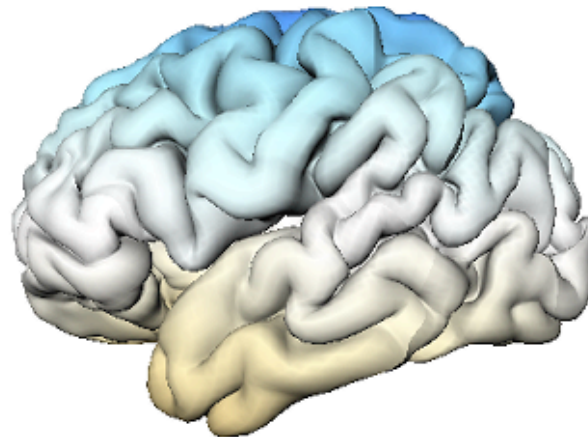
$$v_i(\theta, \varphi) = \sum_{l=0}^k \sum_{m=-l}^l e^{-l(l+1)\sigma} \mu_{lm}^i Y_{lm}(\theta, \varphi) + \epsilon(\theta, \varphi)$$

Coordinates on cortex

Need for a spherical
mapping with the least
metric distortion:



Quasi area preserving algorithm



$T_1, T_2, \dots, T_n$: n triangles in the cortex

$f(T_1), f(T_2), \dots, f(T_n)$: deformed triangles

$\|T_i\|$ area of a triangle.

Flatten the cortex in such a way that

$$\frac{\|f(T_1)\|}{\|T_1\|} = \frac{\|f(T_2)\|}{\|T_2\|} = \dots = \frac{\|f(T_n)\|}{\|T_n\|} = \lambda$$

for some constant λ . There are n equations but approximately $3/2n$ deformation parameters to estimate so there will be infinite number of area-preserving flat maps. Among all solutions, we find one solution that minimize the sum of squared errors:

Quasi area preserving algorithm

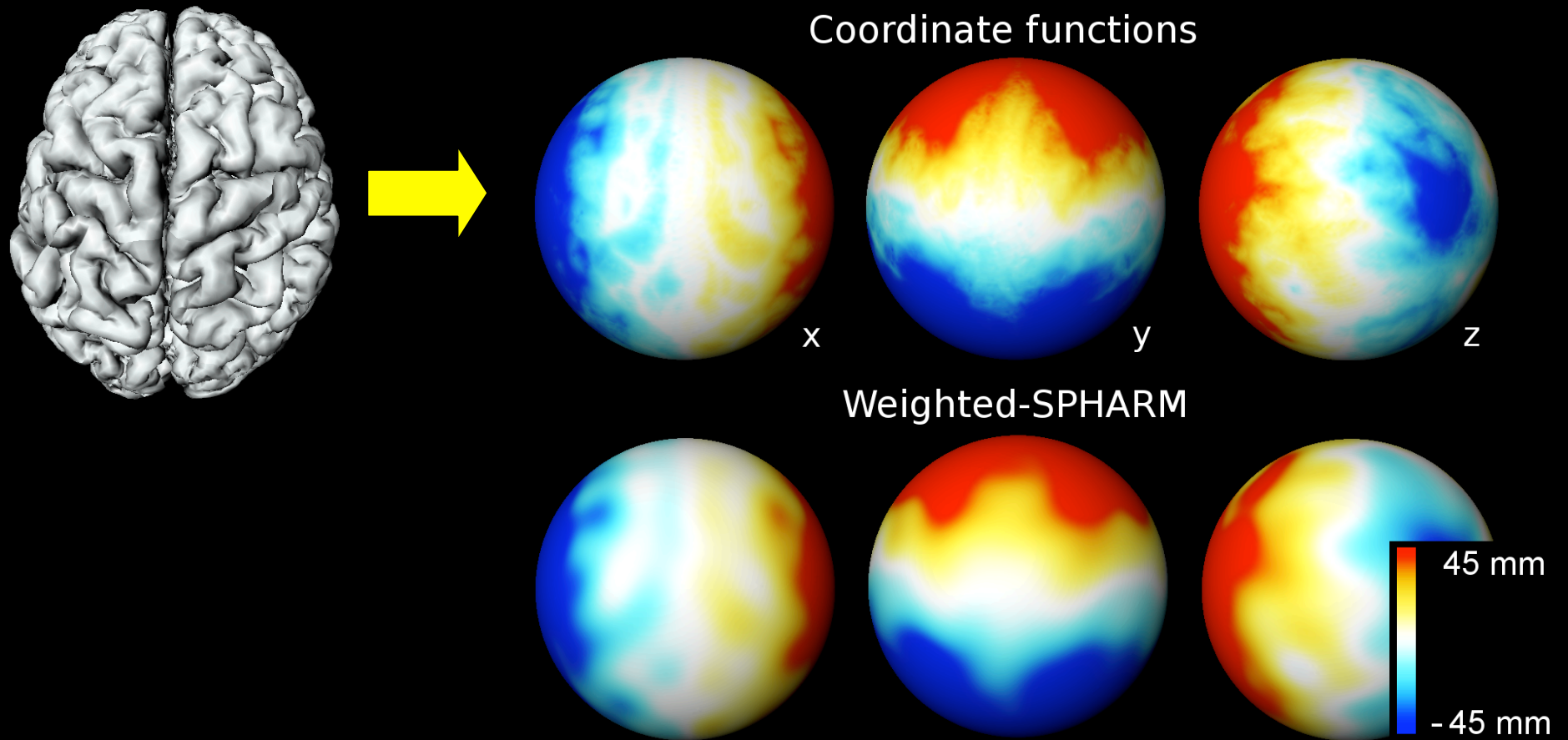
$$L(f) = \sum_{i=1}^n \left(\|T_i\| - \lambda \|f(T_i)\| \right)^2$$

- Gradient descent method is used to find the minimum. In each iteration step, the position of every vertices are slightly moved to locally minimize L until it is not possible to minimize any further.

One possible research project. A student was working on this problem but did not finish it.

Mapping from cortex to unit sphere

Each x , y , z Cartesian coordinates are represented independently.



How do we estimate SPHARM coefficients numerically?

Weighted-SPHARM

$$v_i(\theta, \varphi) = \sum_{l=0}^k \sum_{m=-l}^l e^{-l(l+1)\sigma} f_{lm}^i Y_{lm}(\theta, \varphi)$$

15,000 Fourier coefficients

Iterative residual fitting (IRF) algorithm.

See Chung et al. TMI (2007) for detail.

It will not be discussed here.

Previous approach in estimating Fourier coefficients

- For each point p_i , we have measurement $f(p_i)$.
- Corresponding Fourier series:

$$f(p_i) = \beta_0 \phi_0(p_i) + \beta_1 \phi_1(p_i) + \cdots + \beta_k \phi_k(p_i)$$

- Matrix form:

$$F = \Phi \beta$$

$$\beta = (\Phi' \Phi)^{-1} \Phi' F$$

- This is a nontrivial linear problem

Estimating 15,000 Fourier coefficients

Direct numerical integration takes forever.

Fast Fourier transform (FFT) is not fast either.

Iterative residual fitting (IRF) algorithm

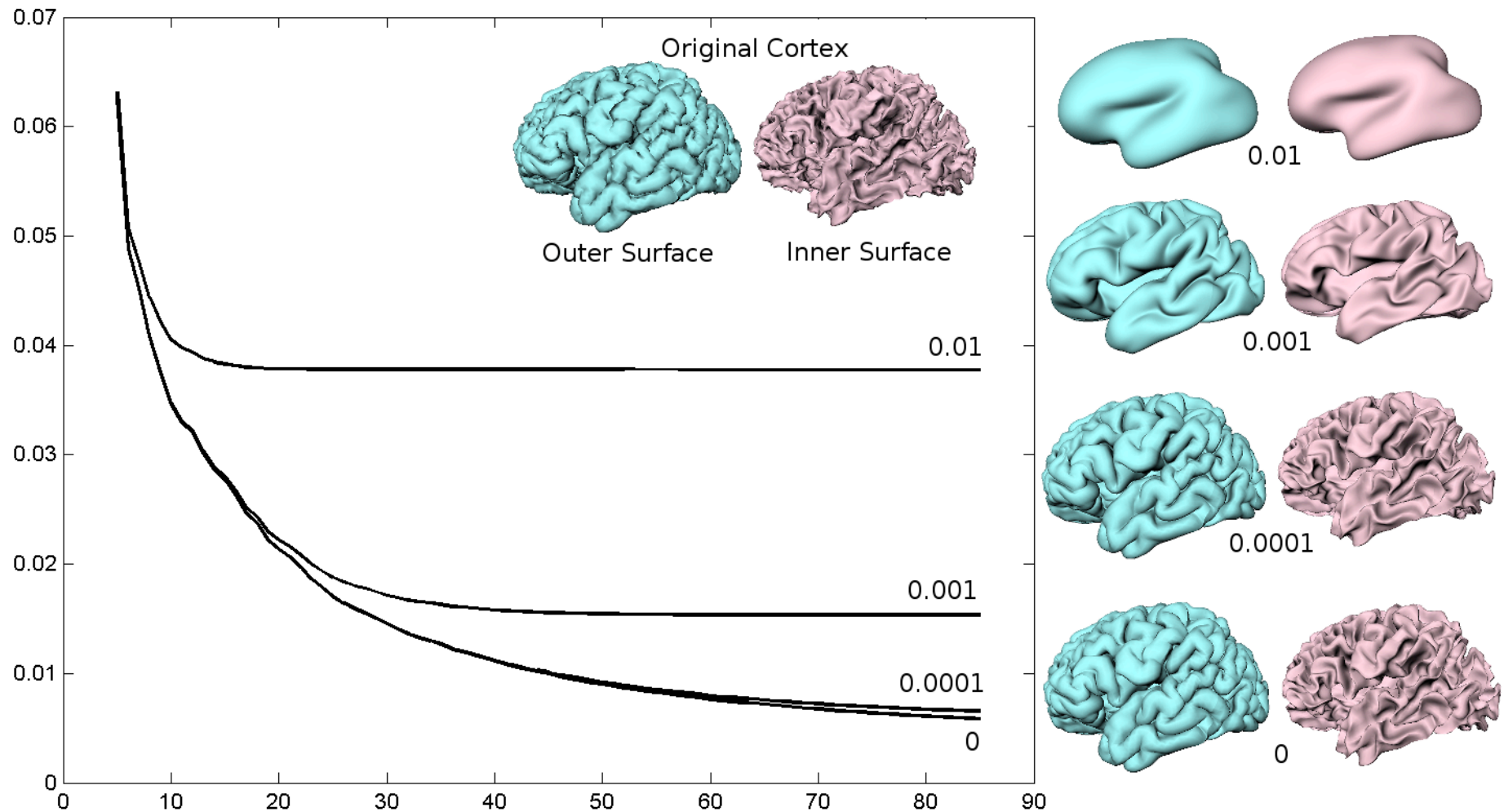
1. Estimate the Fourier coefficients iteratively from lower degree to higher degree.
2. Break one huge linear problem (3GB) into many smaller linear problems (500MB).
3. At each iteration, residual is used to estimate the coefficients of next degree.

Iterative residual fitting (IRF) algorithm

MATLAB implementation can be downloaded from
<http://www.stat.wisc.edu/~mchung/software/weighted-SPHARM/weighted-SPHARM.html>

Sample cortical surface data is also provided.

Weighted-SPHARM at the 80th degree for different bandwidth



Root mean squared error (RMSE)

= error between original surface and weighted-SPHARM

Determining the optimal degree via stepwise forward model selection framework

Consider the following $(k - 1)$ -th degree model

$$f(p_i) = \sum_{l=0}^{k-1} \sum_{m=-l}^l e^{-\lambda(l+1)\sigma} f_{lm} Y_{lm}(p_i) + \epsilon(p_i), \quad i = 1, \dots, n$$

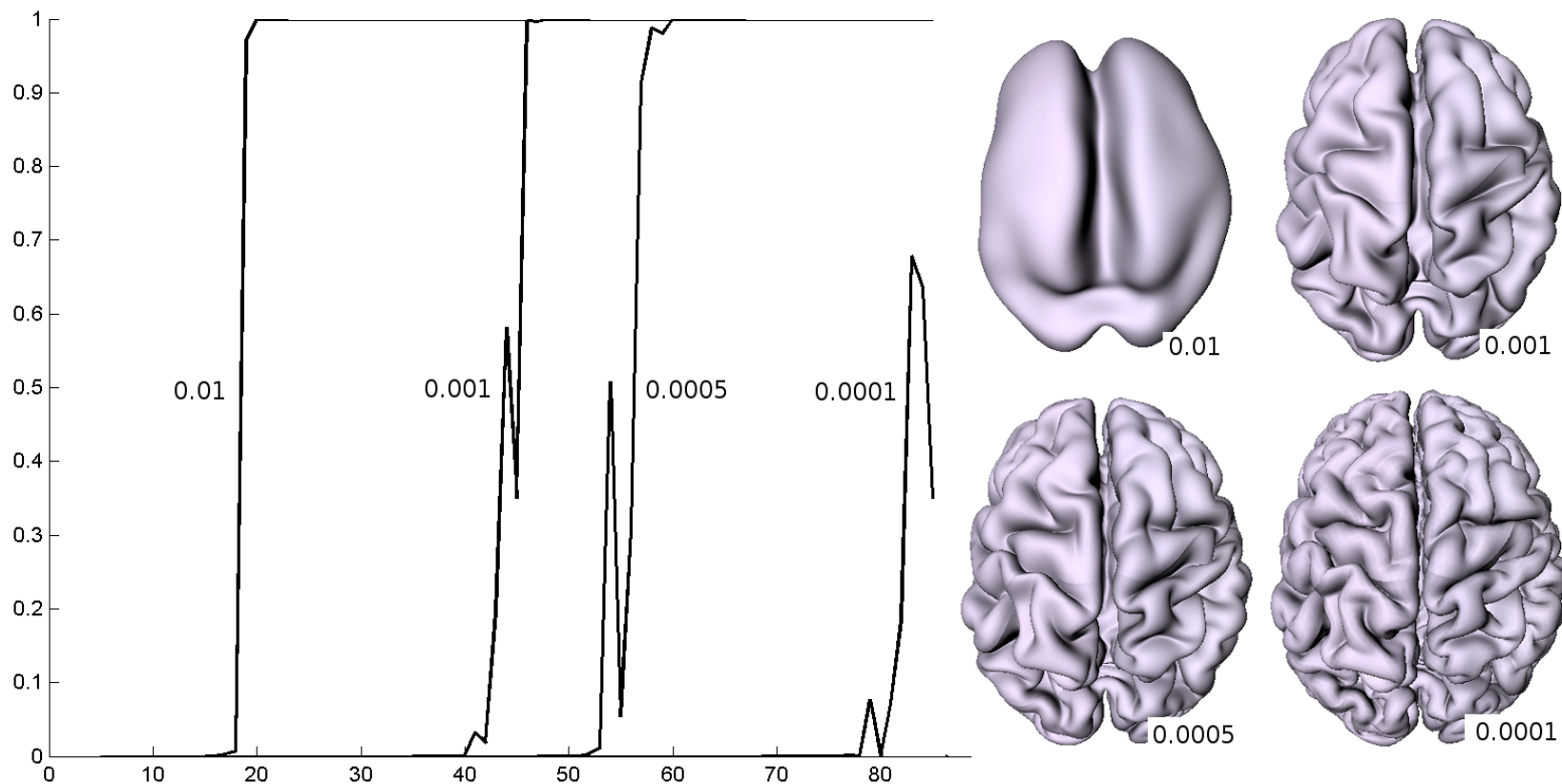
where ϵ are Gaussian random variables. Testing if the k -th degree model is better than the previous $(k - 1)$ -th degree model can be done by testing

$$H_0 : f_{km} = 0 \text{ for all } -k \leq m \leq k.$$

Then under the null hypothesis, the test statistic is

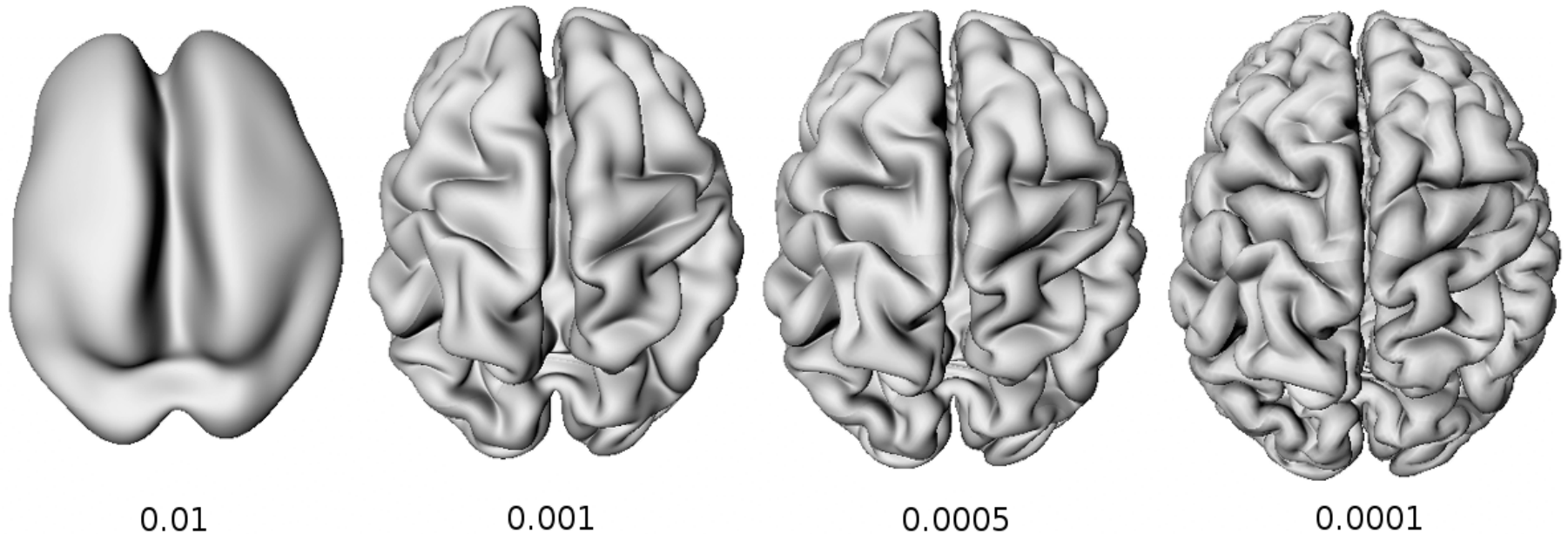
$$F = \frac{(\text{SSE}_{k-1} - \text{SSE}_k)/(2k + 1)}{\text{SSE}_{k-1}/(n - (k + 1)^2)} \sim F_{2k+1, n-(k+1)^2}$$

For each bandwidth σ , optimal degree is automatically selected via **forward best model selection procedure**.



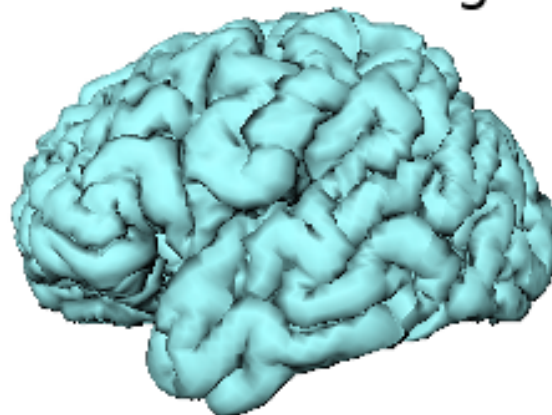
Optimal degree= first P-value >0.05

Weighted-SPHARM at different bandwidth

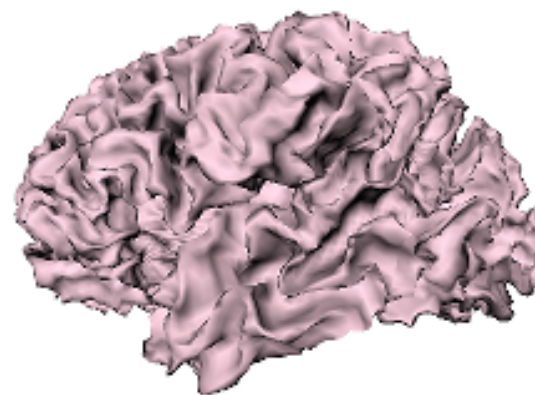


- The degree is selected automatically.
- The only free parameter in the model is the bandwidth.

Original Cortex

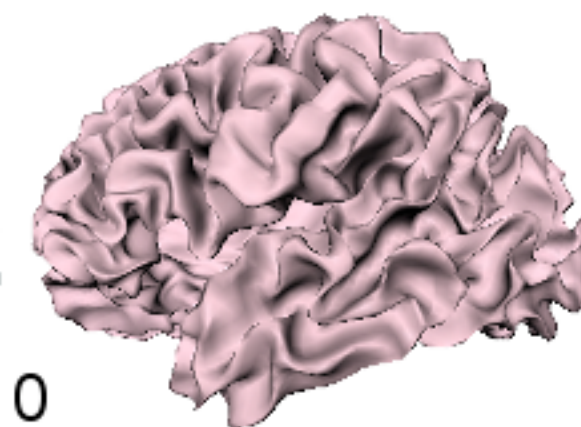
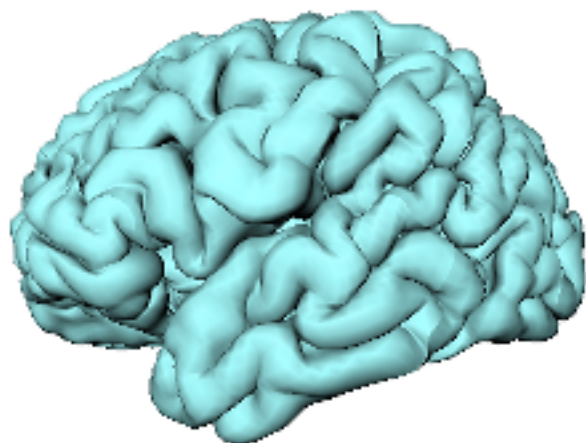


Outer Surface



Inner Surface

80 degree SPHARM

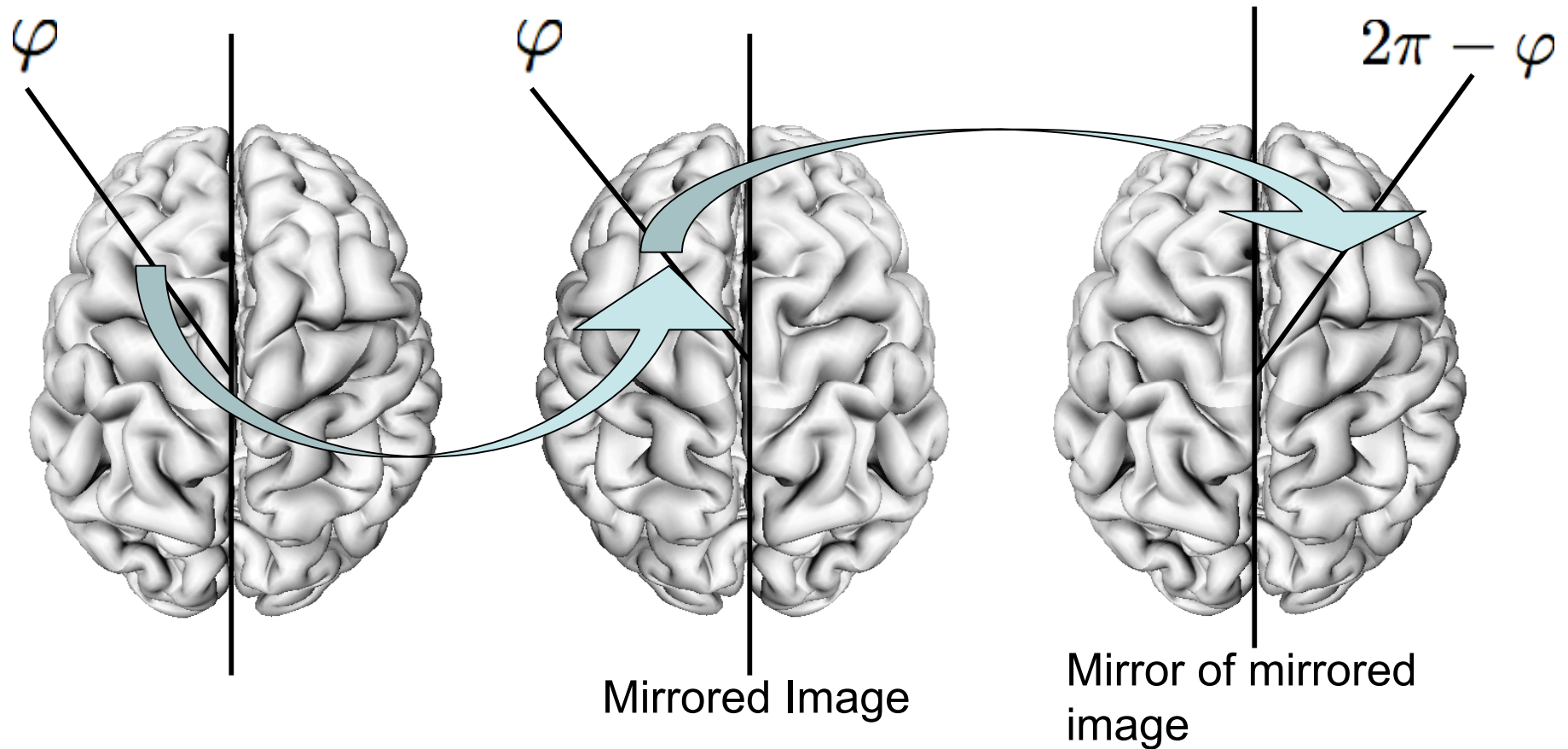


0

Surface registration via SPHARM-correspondence

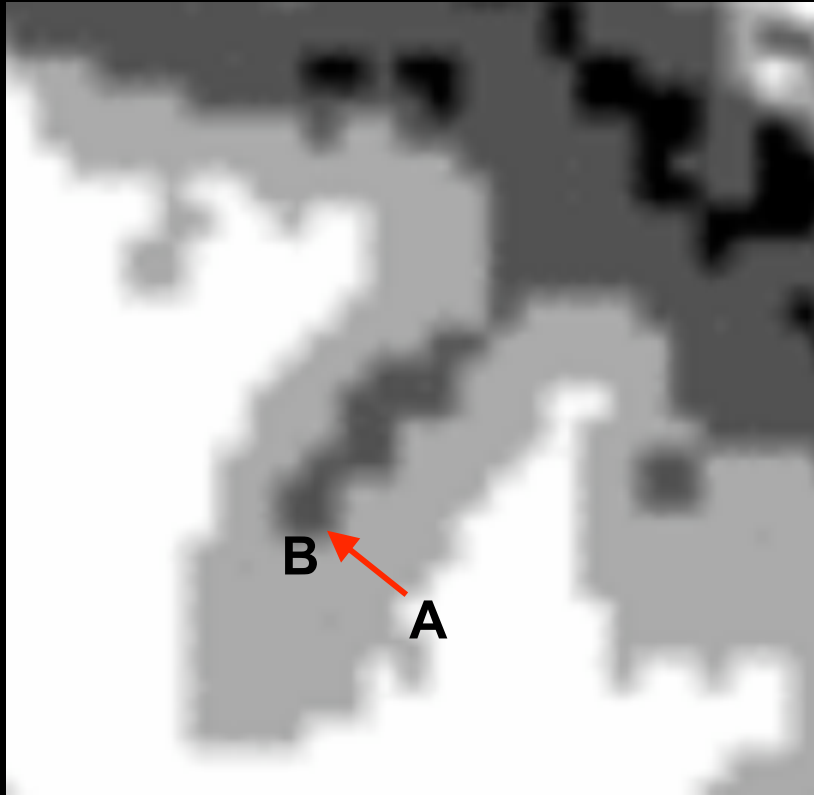
- The correspondence minimizes the sum of squared errors.
- Simply match the SPHARM coefficients.
Most classification techniques are based on this idea.

Establishing Hemispheric correspondence

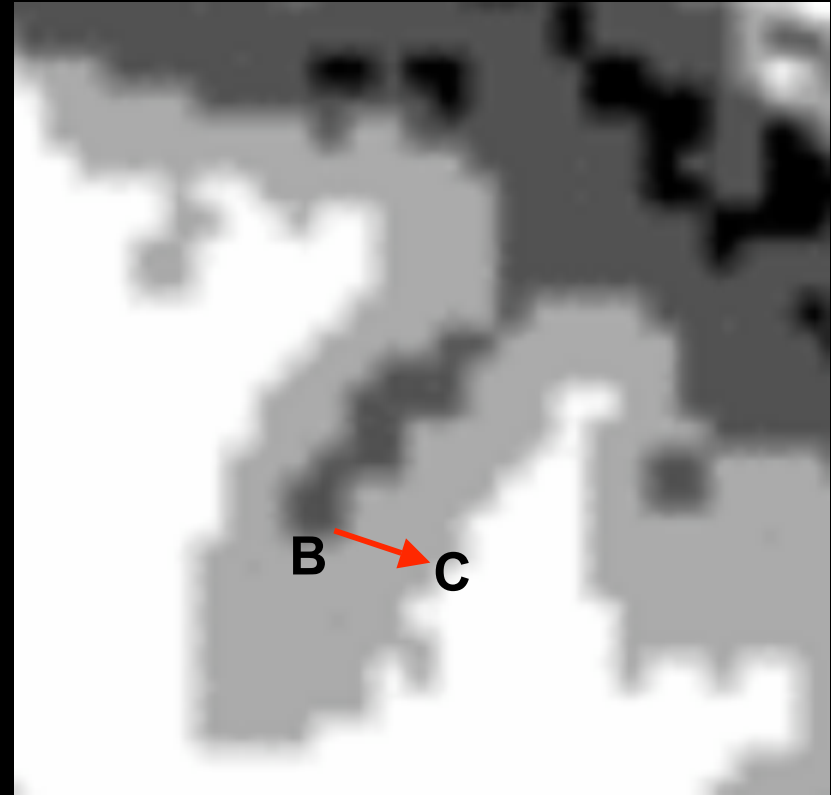


$$Y_{lm}(\theta, 2\pi - \varphi) = \left\{ \begin{array}{ll} -Y_{lm}(\theta, \varphi), & -l \leq m \leq -1, \\ Y_{lm}(\theta, \varphi), & 0 \leq m \leq l. \end{array} \right\}$$

Application 1. Cortical thickness estimation



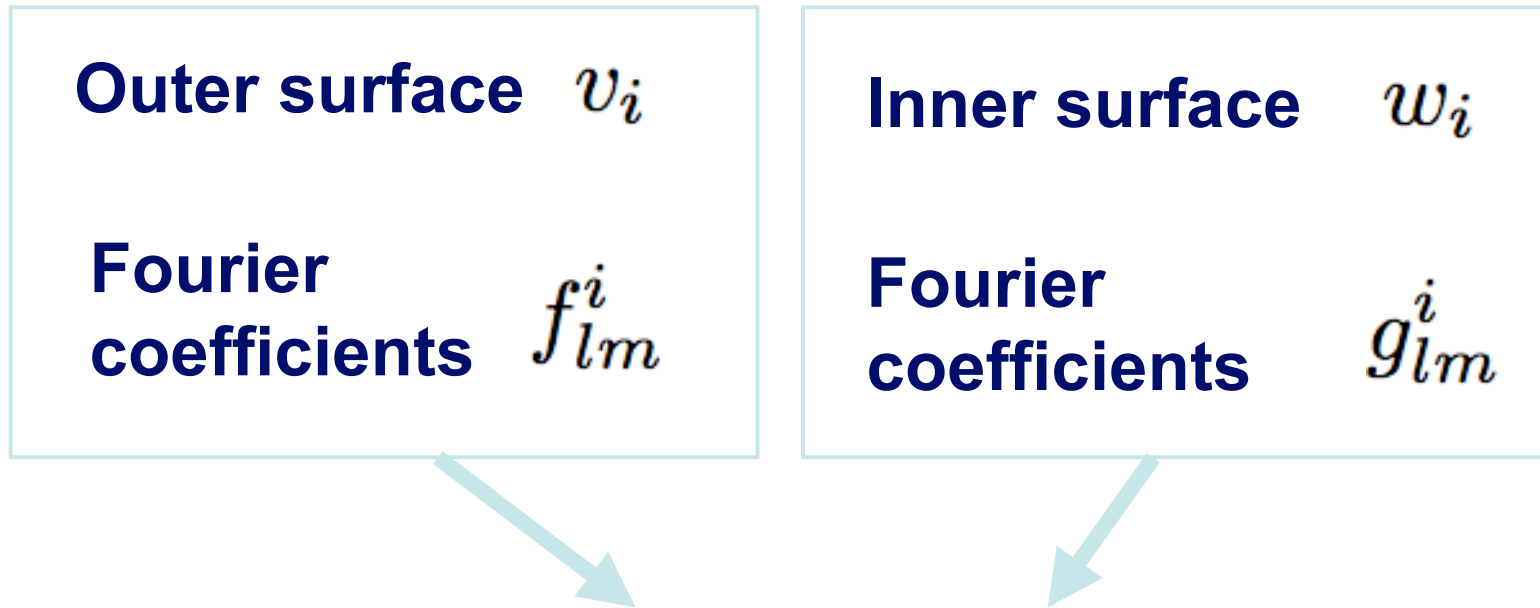
orthogonal projection from A to B



orthogonal projection from B to C

Inconsistency in defining cortical thickness.

Consistent definition using weighted-SPHARM

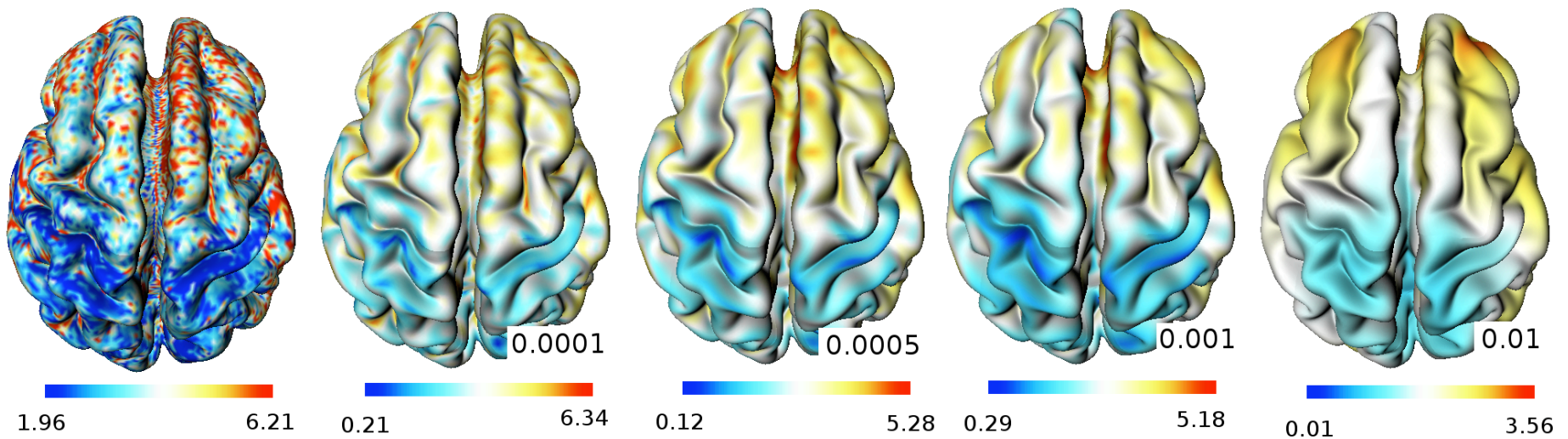


Cortical thickness

$$\sum_{l=0}^k \sum_{m=-l}^l e^{-\lambda(\lambda+1)\sigma} \left[\sum_{i=1}^3 (g_{lm}^i - f_{lm}^i)^2 \right]^{1/2}$$

MIAR (2006), TMI (2007)

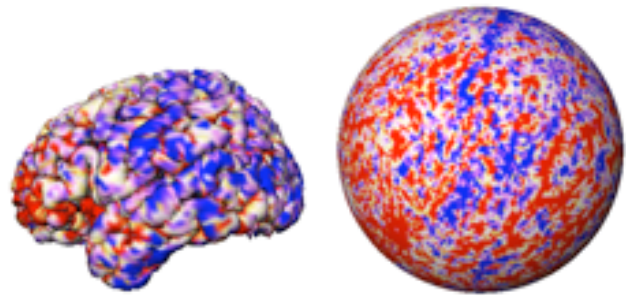
SPHARM estimation of cortical thickness



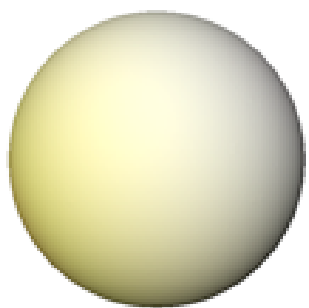
Thickness
estimation
based on
traditional
method

Too much
smoothing

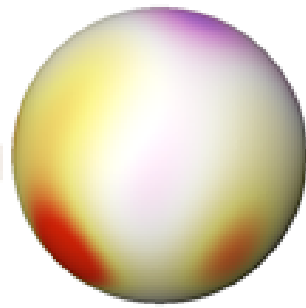
Weighted-SPHARM of cortical thickness



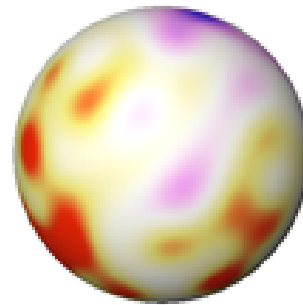
$$\sum_{l=1}^k e^{-l(l+1)\sigma} \sum_{m=-l}^l f_{lm} Y_{lm}$$



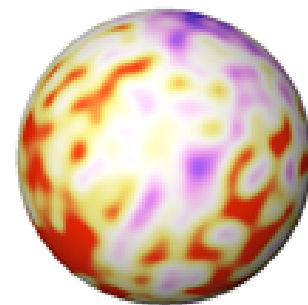
k=1



7



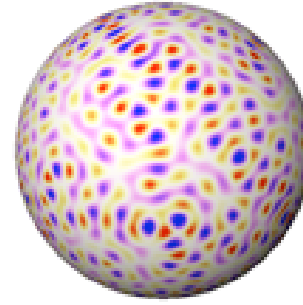
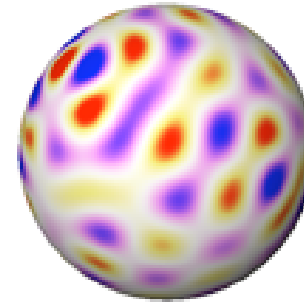
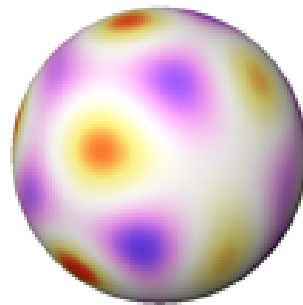
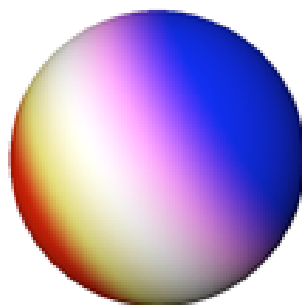
14



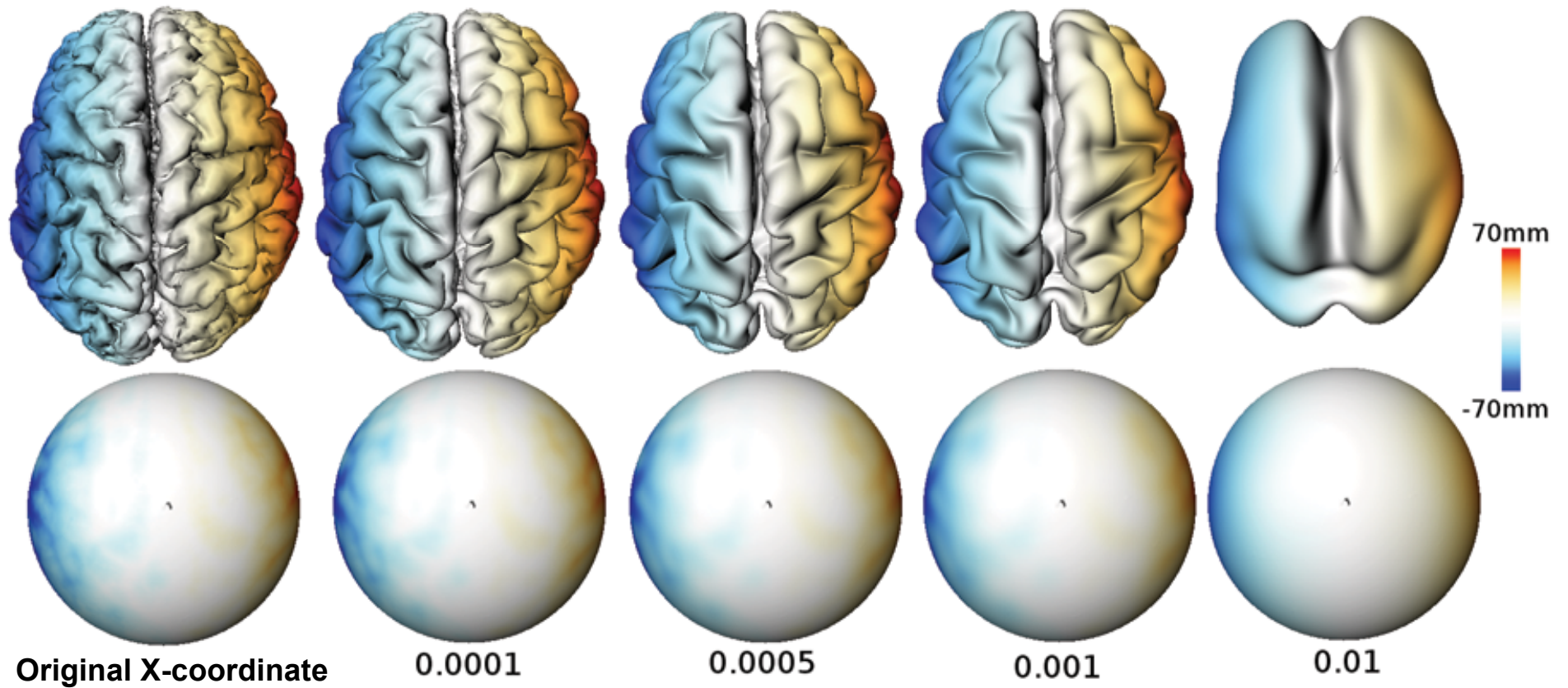
42



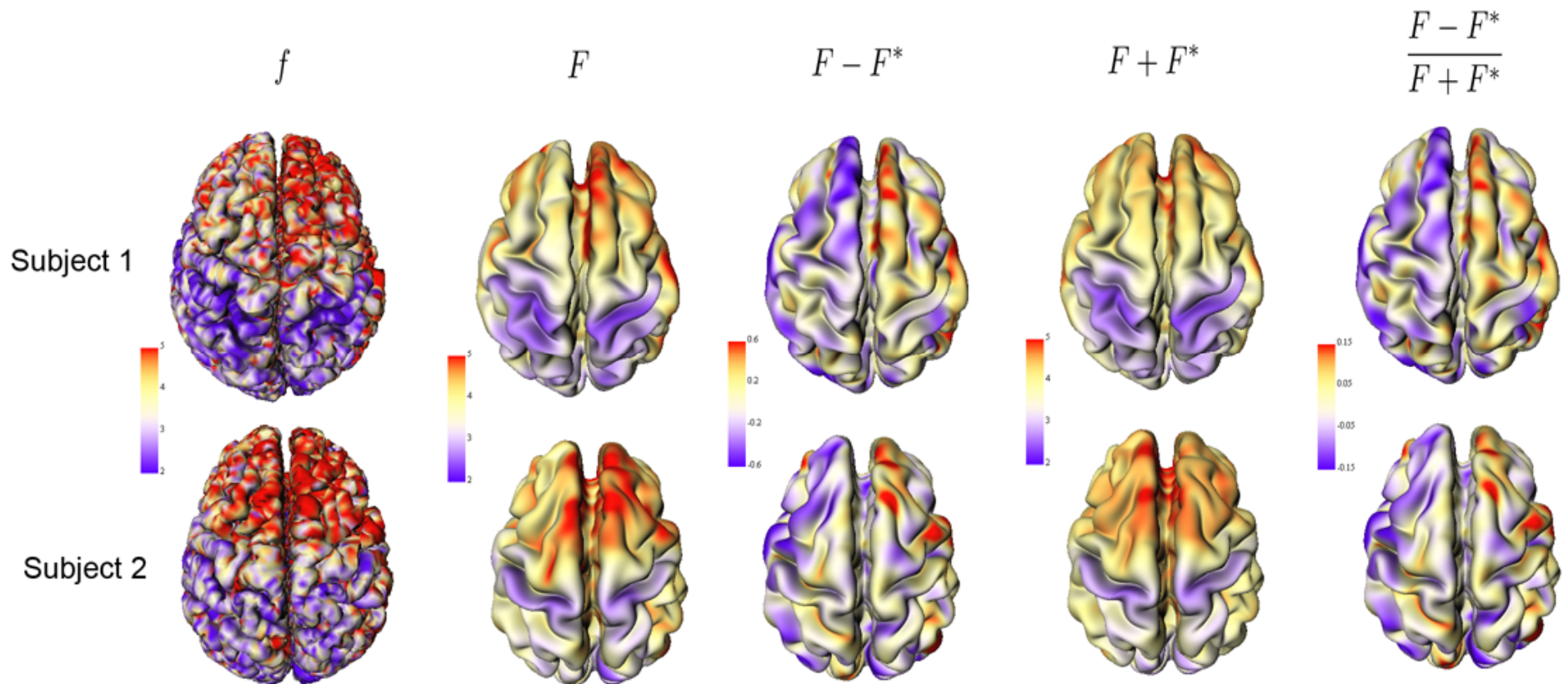
$$\sum_{m=-k}^k f_{km} Y_{km}$$



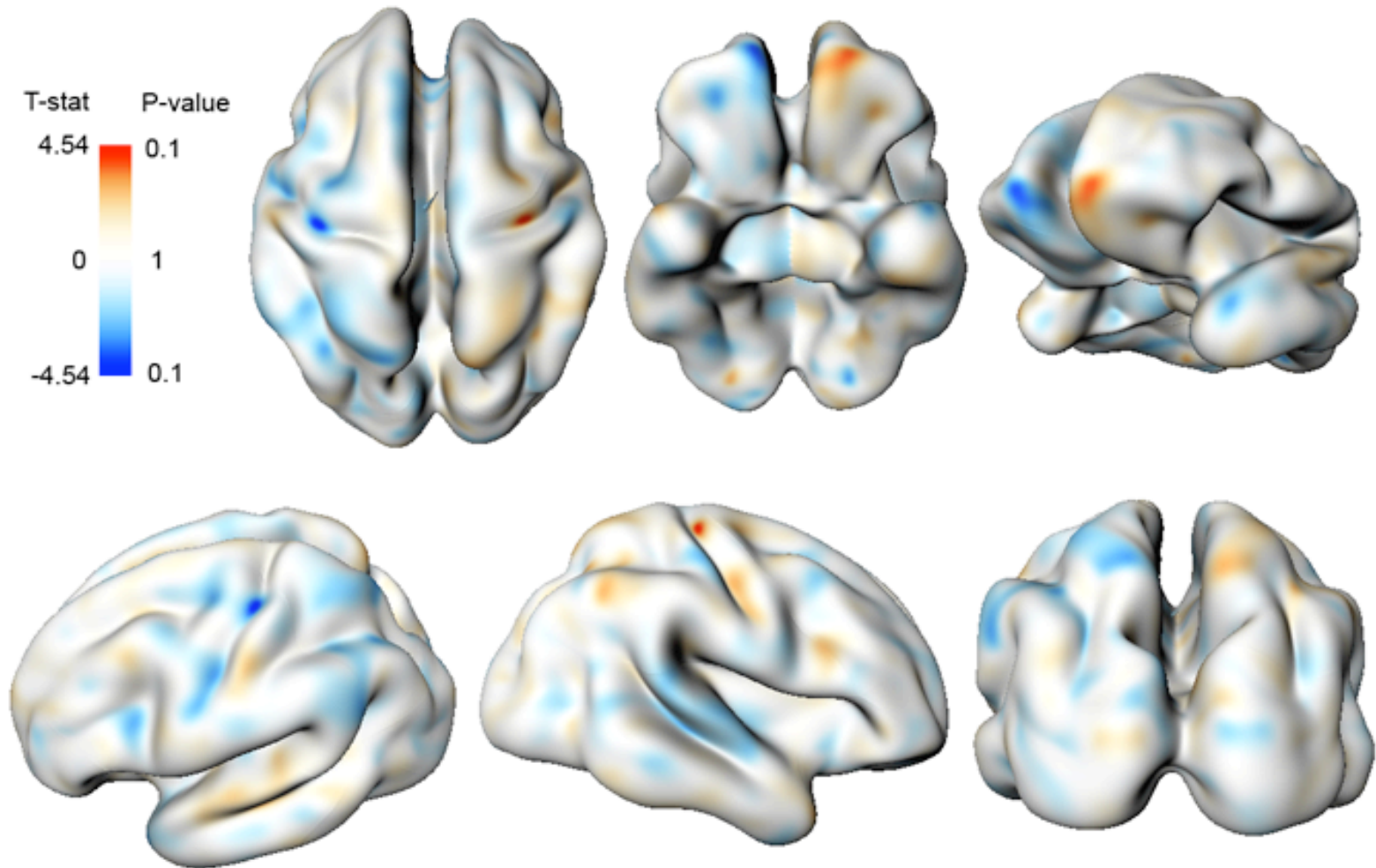
Weighted-SPHARM at different scale



Cortical Asymmetry Index



Cortical Asymmetry Result



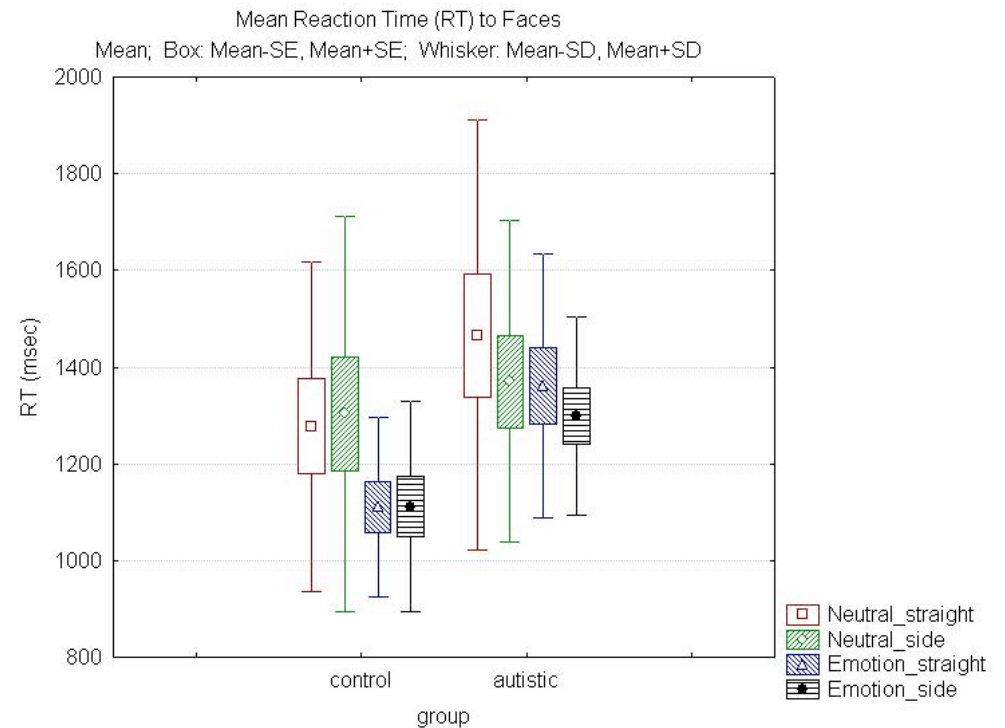
Application: correlating anatomy with nonimaging measures

Facial emotion discrimination task response time

2 (Emotion) $\times$ 2 (Orientation)

Neutral

Emotional



Dalton et al. (2005)

24 emotional faces, 16 neutral faces

Let $Y = (Y_1, Y_2)$ be two variables of interests and $X = (X_1, \dots, X_p)$ be a row vector of multivariate that should be removed. For instance, we may let Y_1 to be the cortical thickness and Y_2 be the face recognition tasks while X_1 and X_2 are age and total surface area. The covariance matrix of $(Y, X)'$ is given by

$$\mathbb{V}(Y, X)' = \begin{pmatrix} \Sigma_{YY} & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{XX} \end{pmatrix} \quad (1)$$

Note Σ_{YY} , Σ_{YX} and Σ_{XX} are the cross covariance matrices. Then the partial covariance of Y given X is

$$\Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY} = (\sigma_{ij})$$

The *partial correlation* $\rho_{Y_i, Y_j|X}$ is then defined as the correlation between variables Y_i and Y_j while controlling for other variables X and given by

$$\rho_{Y_i, Y_j|X} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}.$$

The reason we are using the *conditional* notation $|$ in defining the partial correlation is due to the fact that the partial correlation is equivalent to *conditional correlation* if $\mathbb{E}(Y|X) = a + BX$ for some vector a and matrix B , which is true under normal assumption. This is the formulation we used to compute the par-

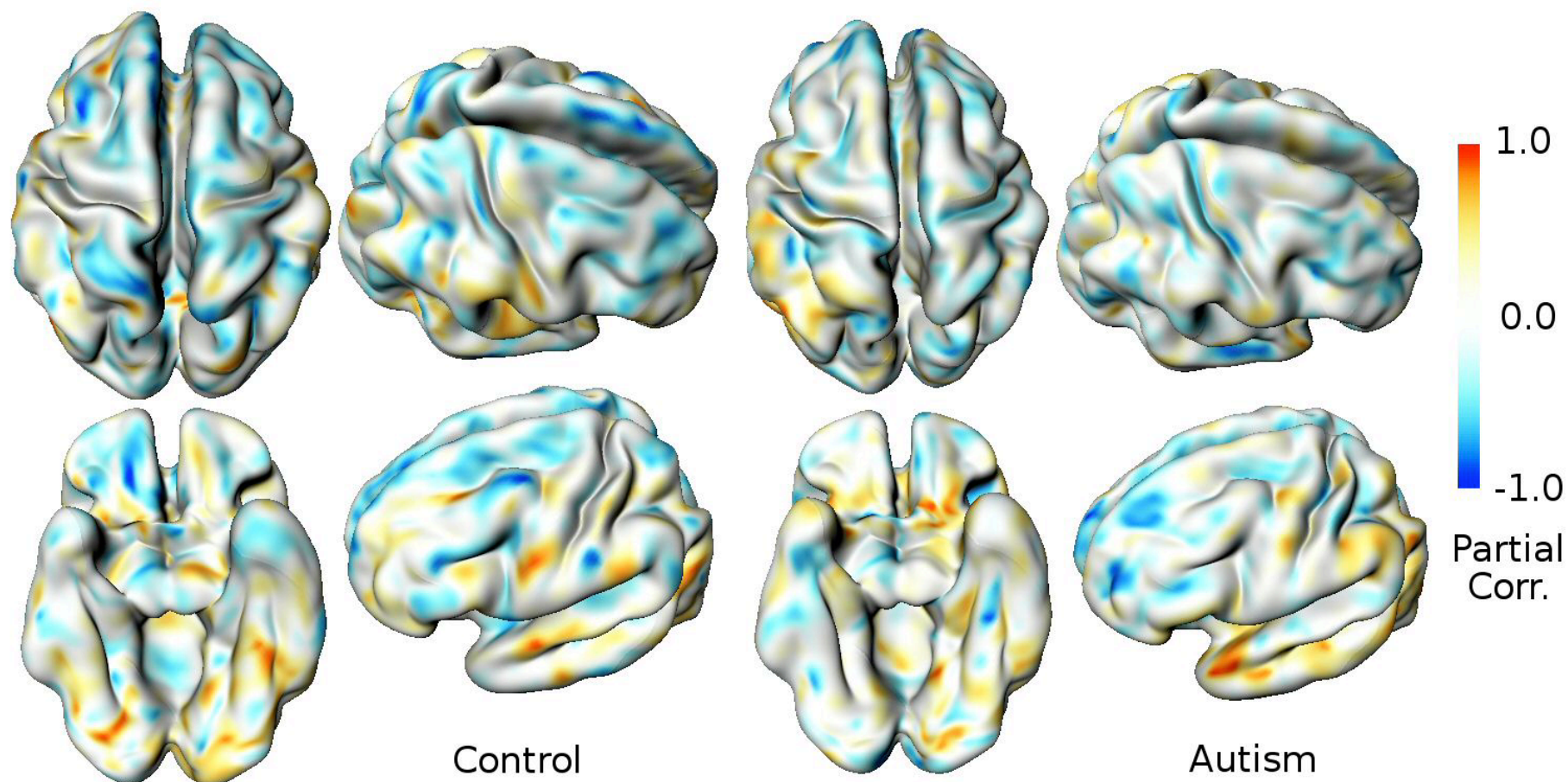
See

<http://www.stat.wisc.edu/~mchung/papers/partial.correlation.TR1109.2005.pdf>

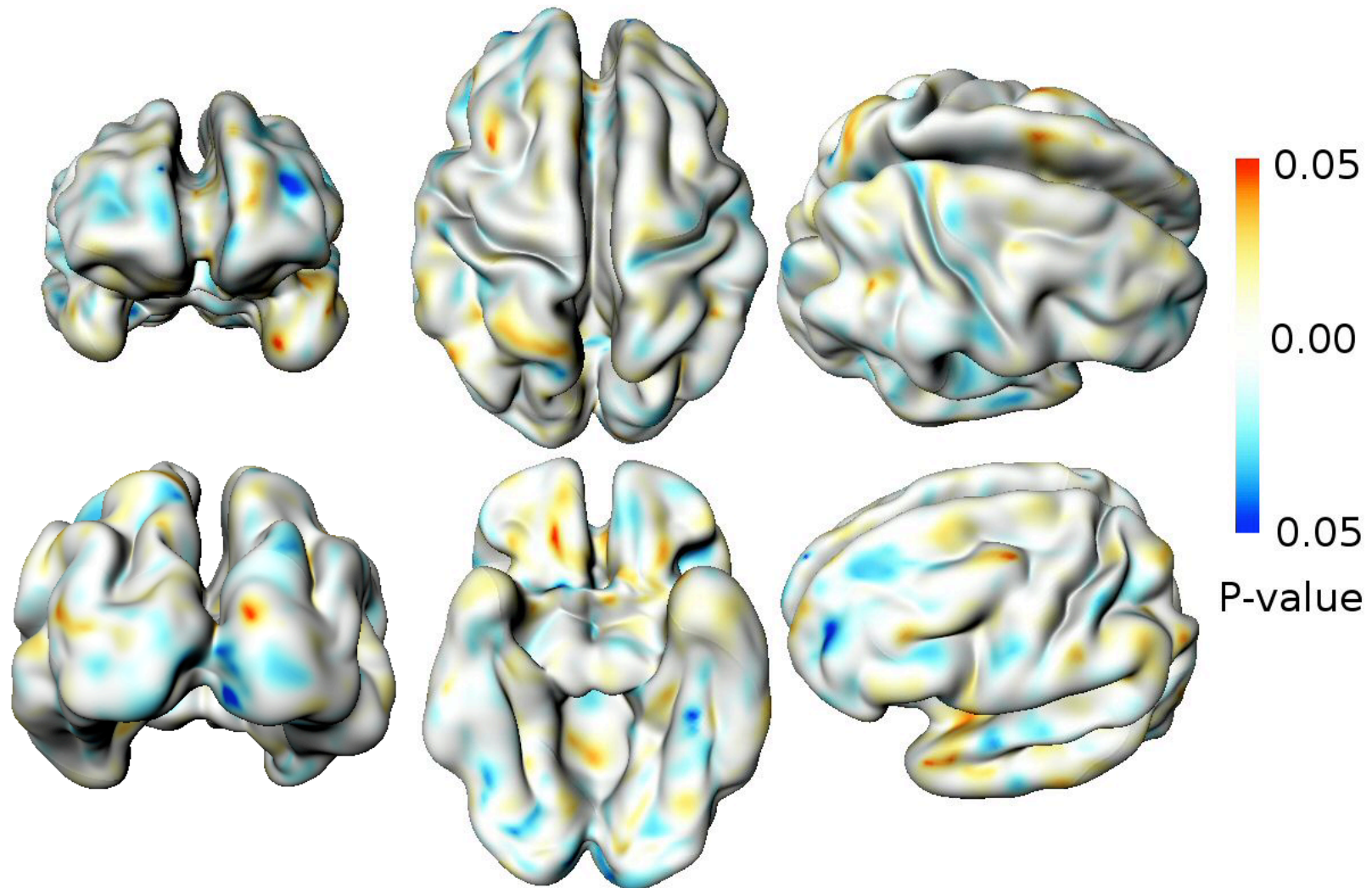
detailed explanation
on partial correlation

Correlation between response time and cortical thickness

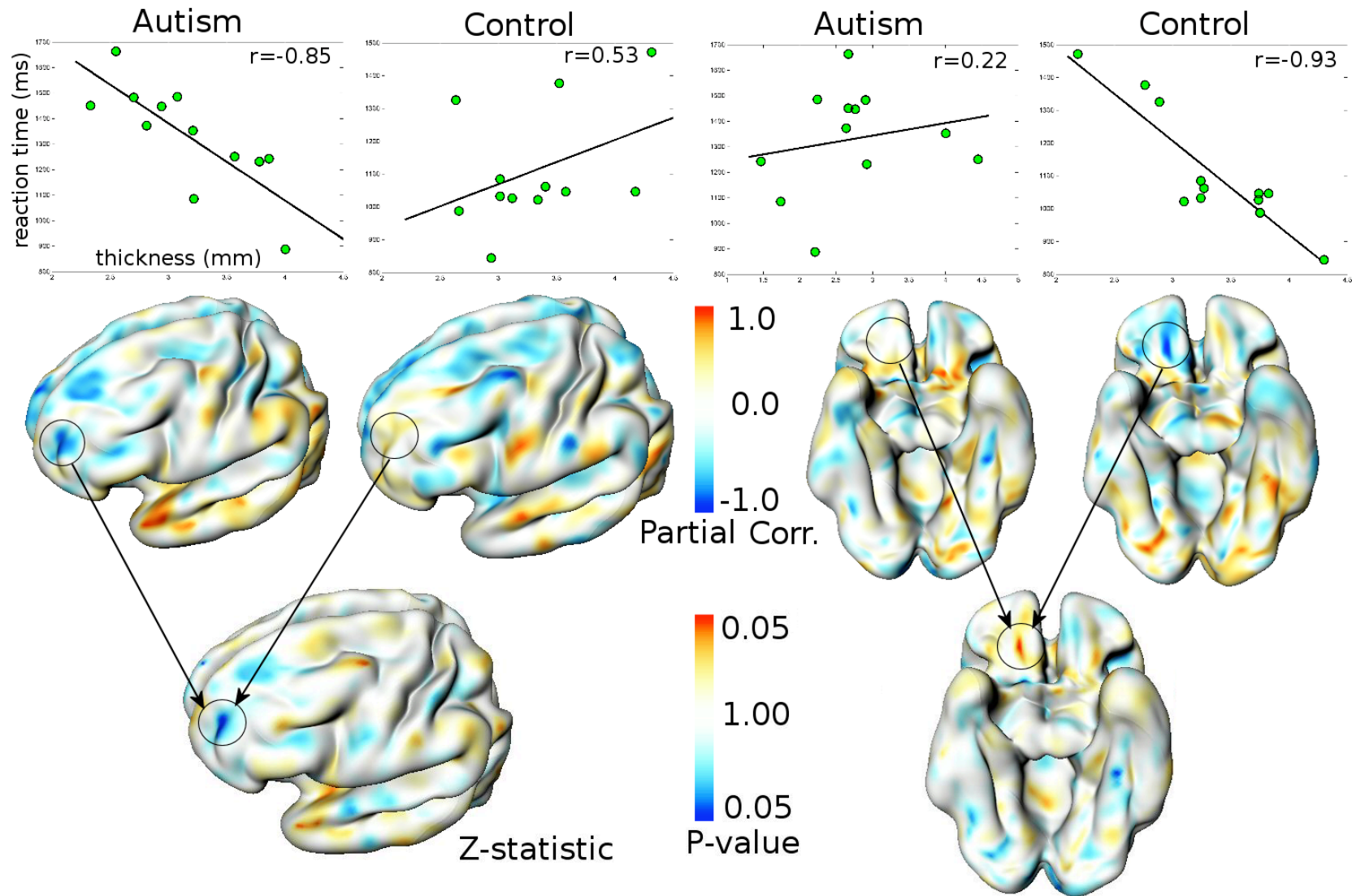
(the effect of age and brain size difference has been removed)



The final SPM showing statistically significant correlation difference between the groups.

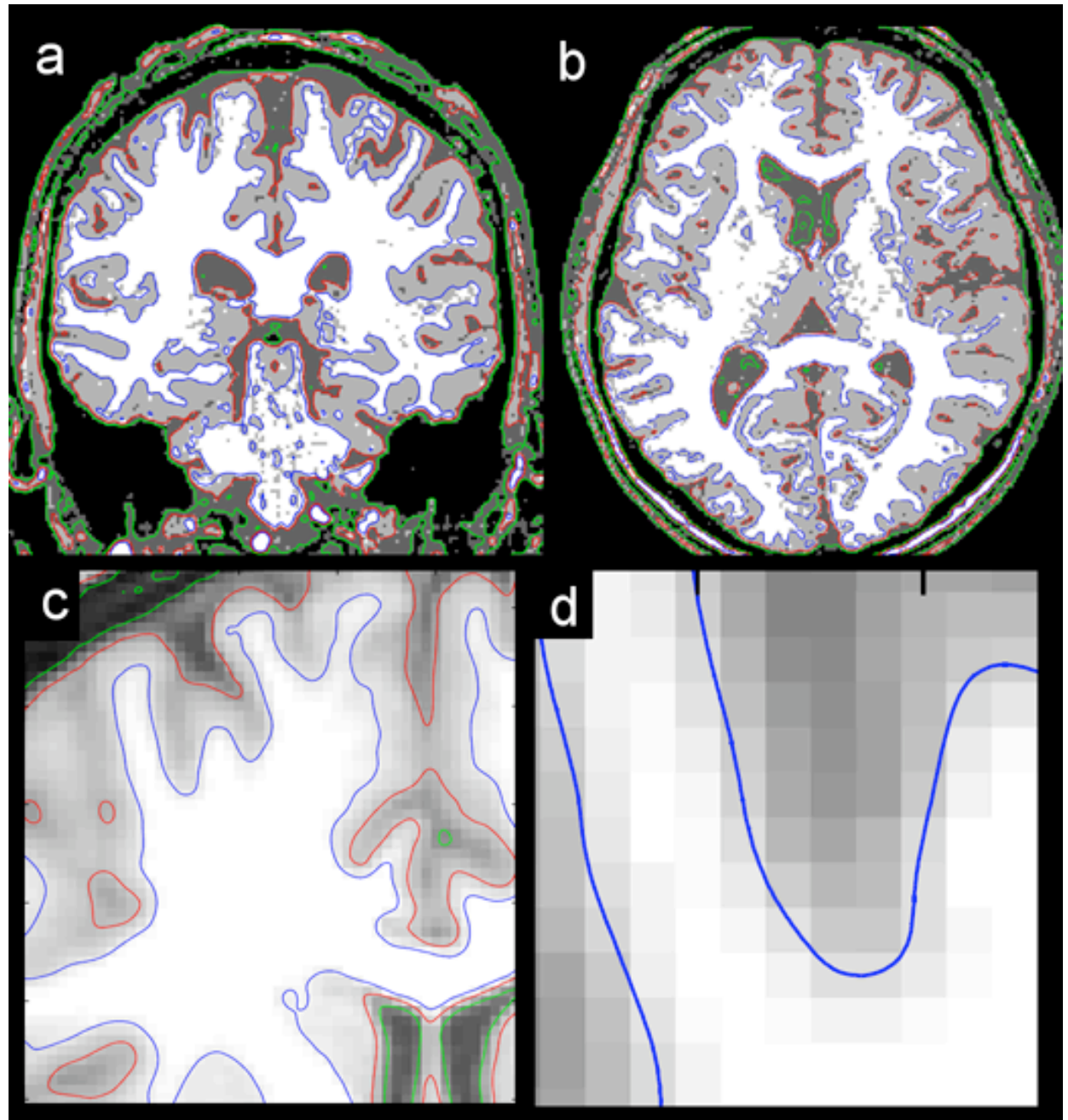


Scatter plot of response time vs. thickness



Thin Plate Spline (TPS) segmentation and modeling

TPS represents anatomical boundary as the zero level set of smooth function consists of polynomial and radial basis functions (Wahba, 1990; Xie et al., 2005a).



Thin Plate Spline

Measurement f is represented as

$$f(p) = \sum_i \alpha_i \phi_i(p) + \sum_j \beta_j \varphi(p - p_j)$$

where ϕ_i is polynomial basis and φ is the TPS radial basis

Parameters are estimated by minimizing

$$\min_f \sum_{i=1} |y_i - f(p_i)|^2 + \lambda J_3^2(f),$$

Lecture 7 Topics

The effect of SNR to VBM

Logistic Discriminant Analysis