

Medical Image Analysis

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Lecture 11.

Anisotropic Smoothing

March 05, 2007

Motivation for spatially adaptive anisotropic smoothing

Need for smoothing data while preserving boundary information

Differential equation models for statistical functions¹

James O. RAMSAY

238

RAMSAY

Vol. 28, No. 2

Wealth concentration
in Montreal area

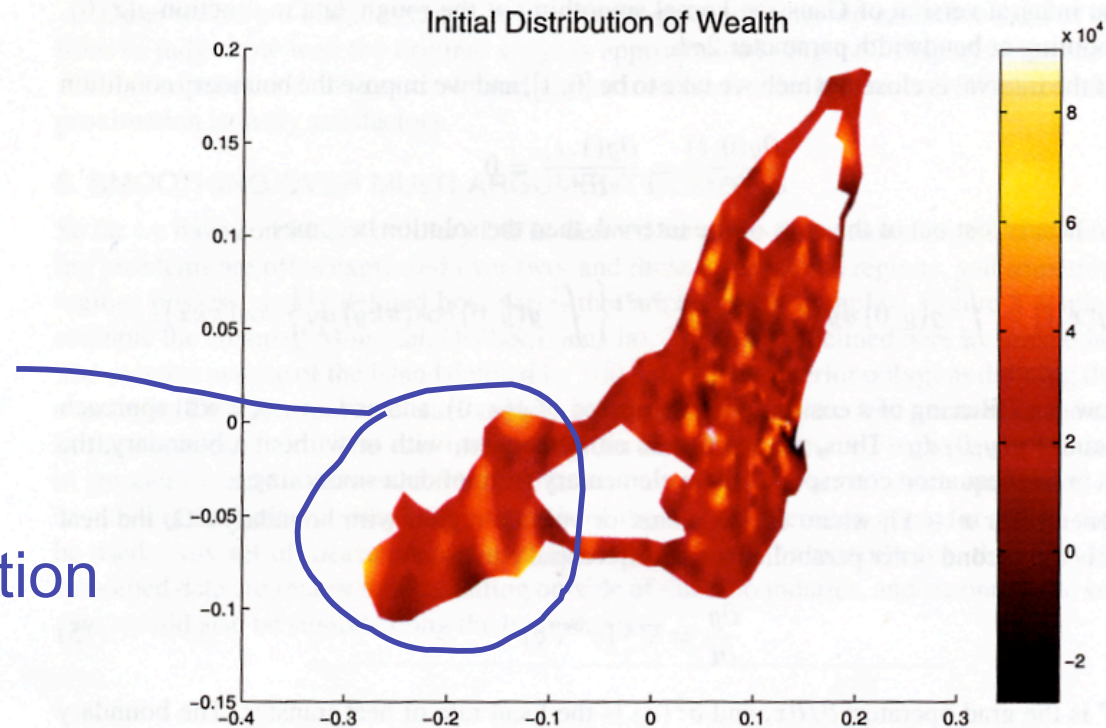
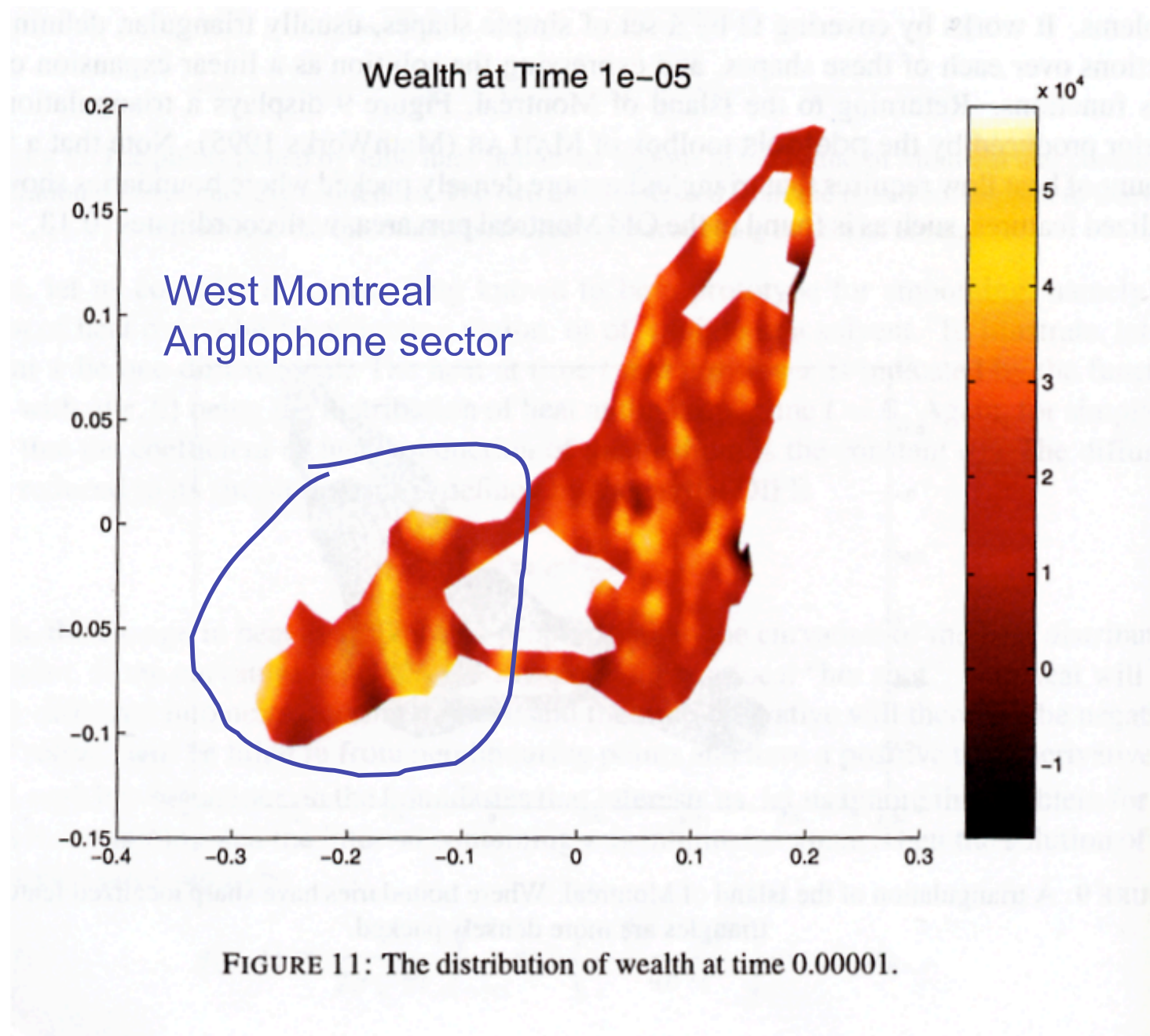


FIGURE 10: The distribution of wealth on the Island of Montréal. This defines the initial state $g(x, 0)$ of the system described by partial differential equation (15) and boundary condition (16).



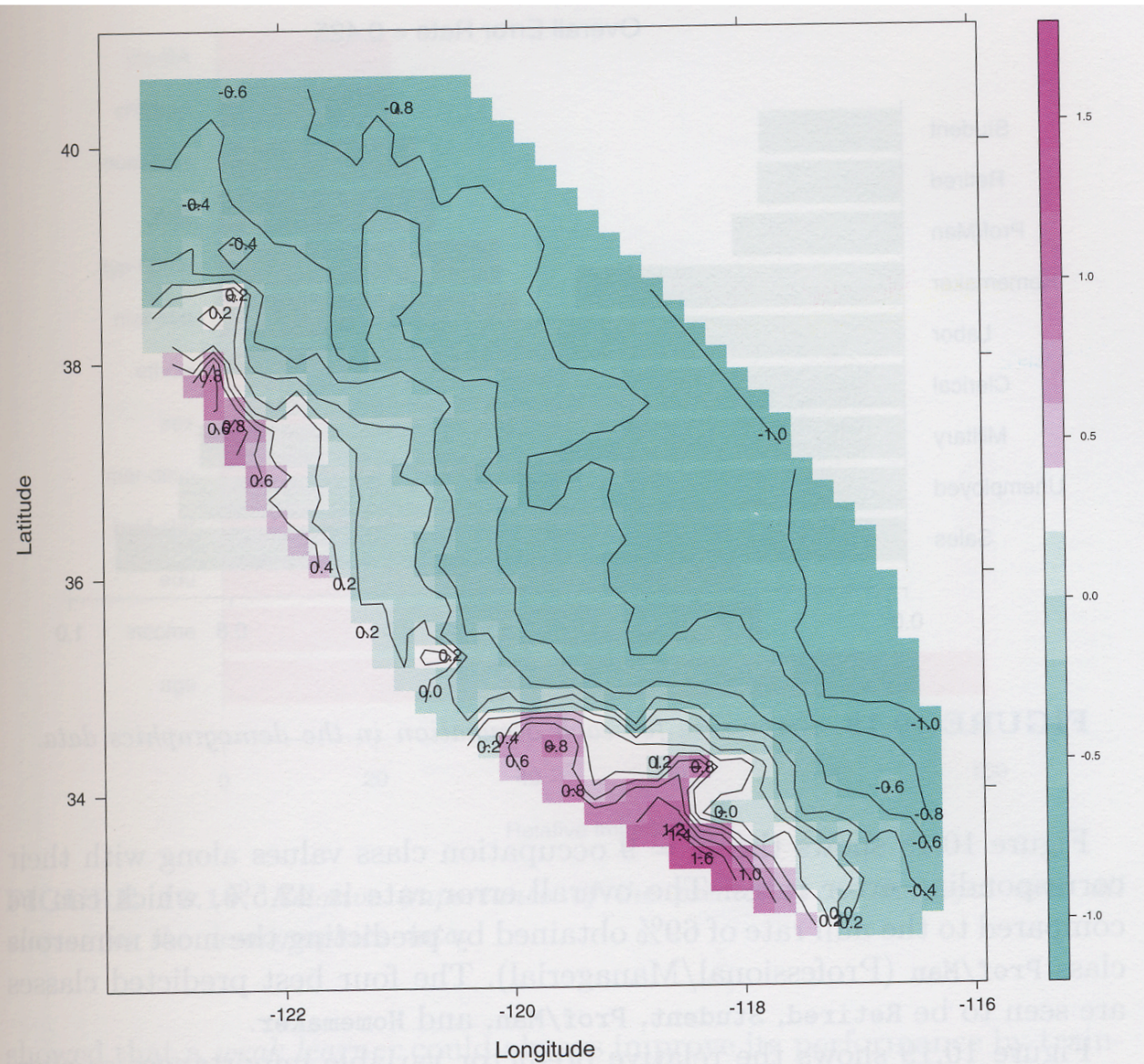


FIGURE 10.17. *Partial dependence of median house value on location in California.*

Source:
Hastie

Smoothing while preserving edge



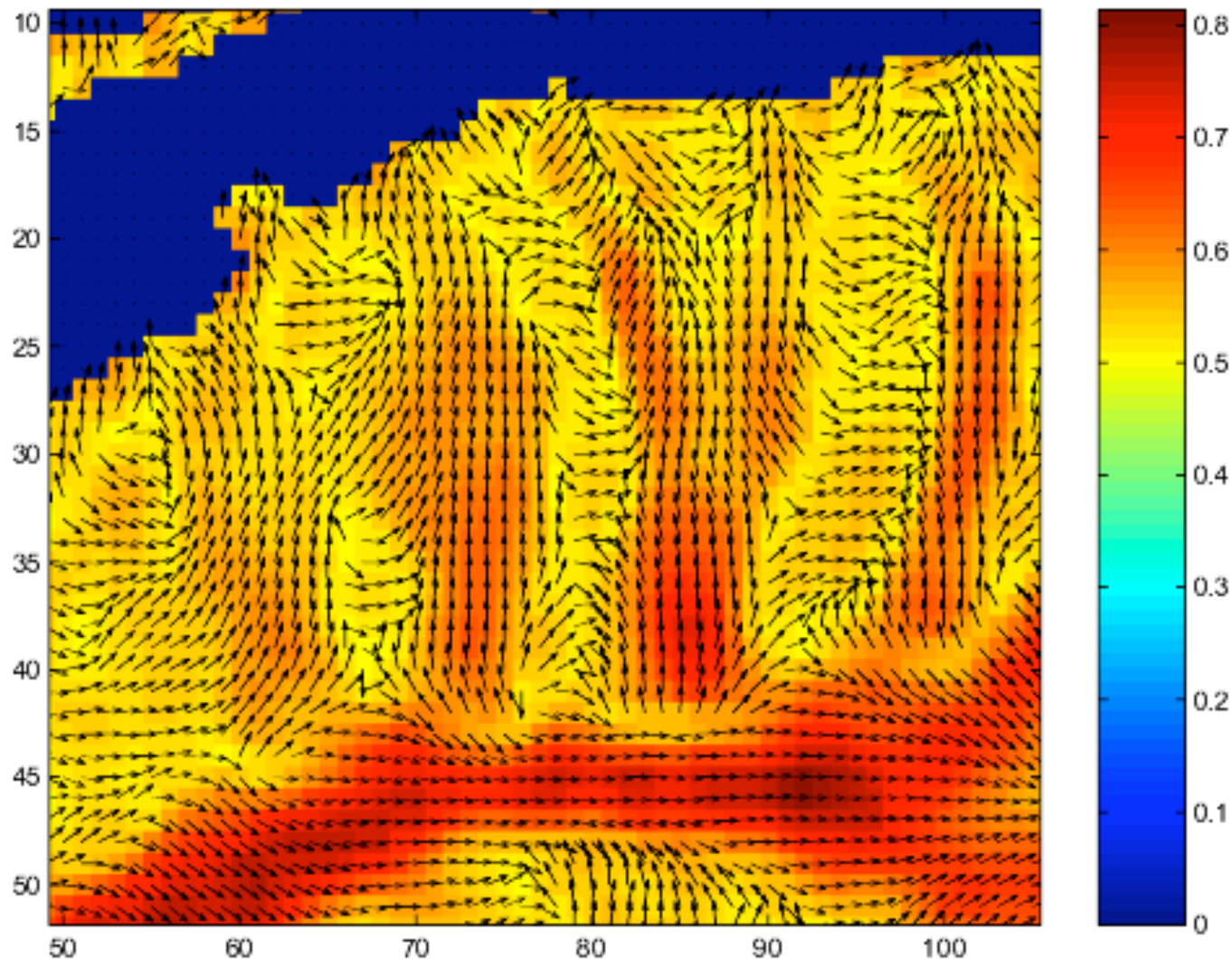
NOISY IMAGE



DENOISED IMAGE

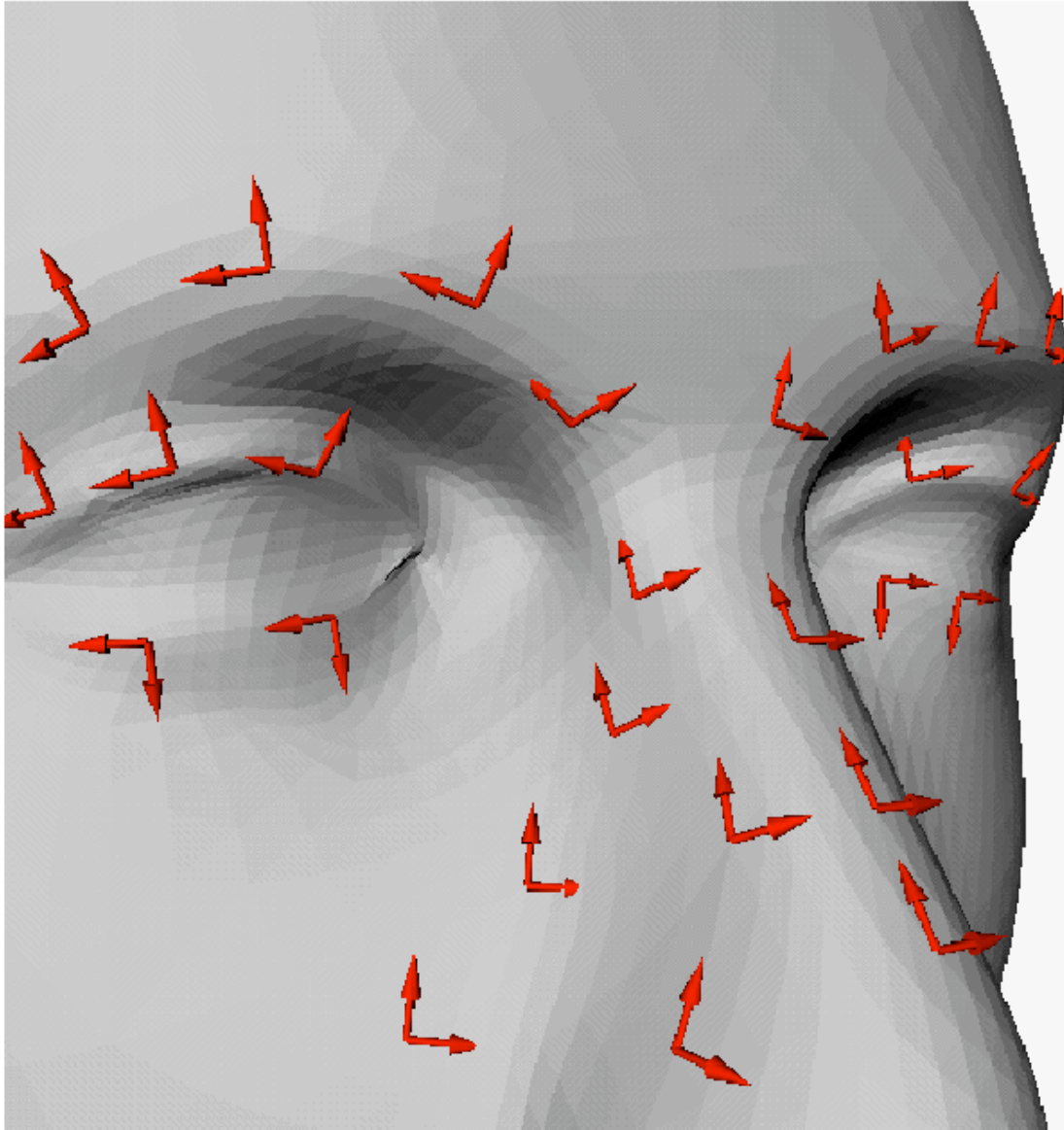
Figure 10. Image denoising using Beltrami flow [Kimmel et al].

Diffusion tensor imaging (DTI): Smoothing along vector or tensor fields



Arrows = Principal eigenvectors
Colors = Principal eigenvalues of diffusion coefficient matrix.

Smoothing along tensor fields



Principal curvature direction
Meyer *et al.*

Two main approaches

- Anisotropic kernel smoothing
- Anisotropic diffusion equation

Kernel smoothing

- Isotropic kernel smoothing: weighted averaging where weights are isotropic
- Anisotropic kernel smoothing: weights are not isotropic. It contains directional information

Anisotropic kernel

$$\mathbf{x} = (x_1, \dots, x_n)' \in \mathbb{R}^n$$

n -dimensional isotropic Gaussian kernel

$$K(\mathbf{x}) = \exp(-\mathbf{x}'\mathbf{x}/2)/(2\pi)^{n/2}$$

anisotropic Gaussian kernel

$$K_H(x) = K(H^{-1}\mathbf{x})/\det(H)$$

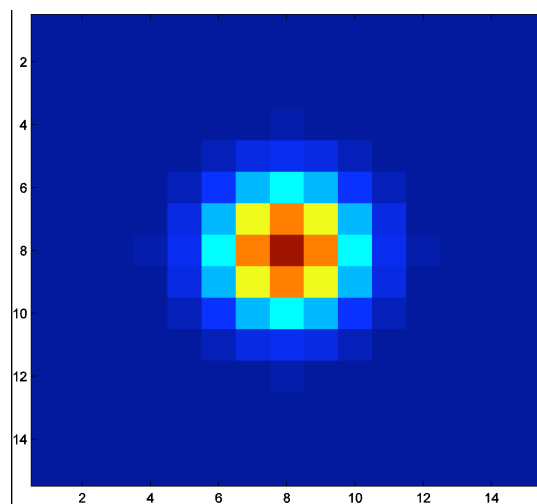
H bandwidth matrix

H H' covariance matrix

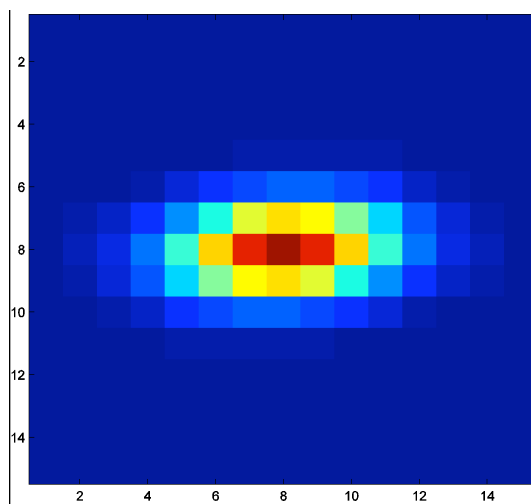
Anisotropic Gaussian kernel smoothing

$$F(\mathbf{x}) = K_H * f(\mathbf{x})$$

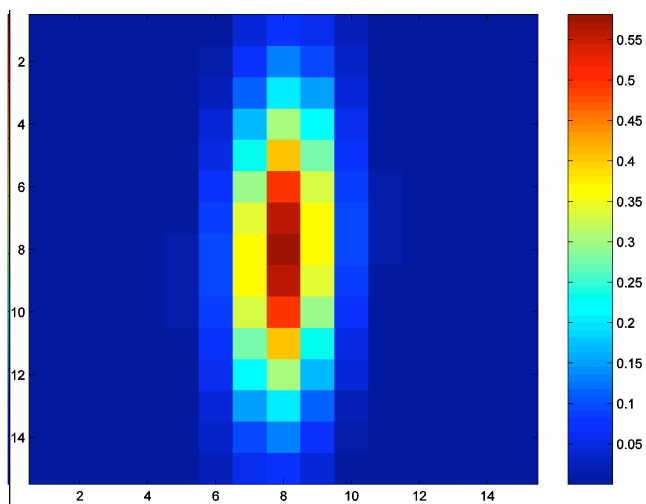
Kernel weights



Isotropic kernel

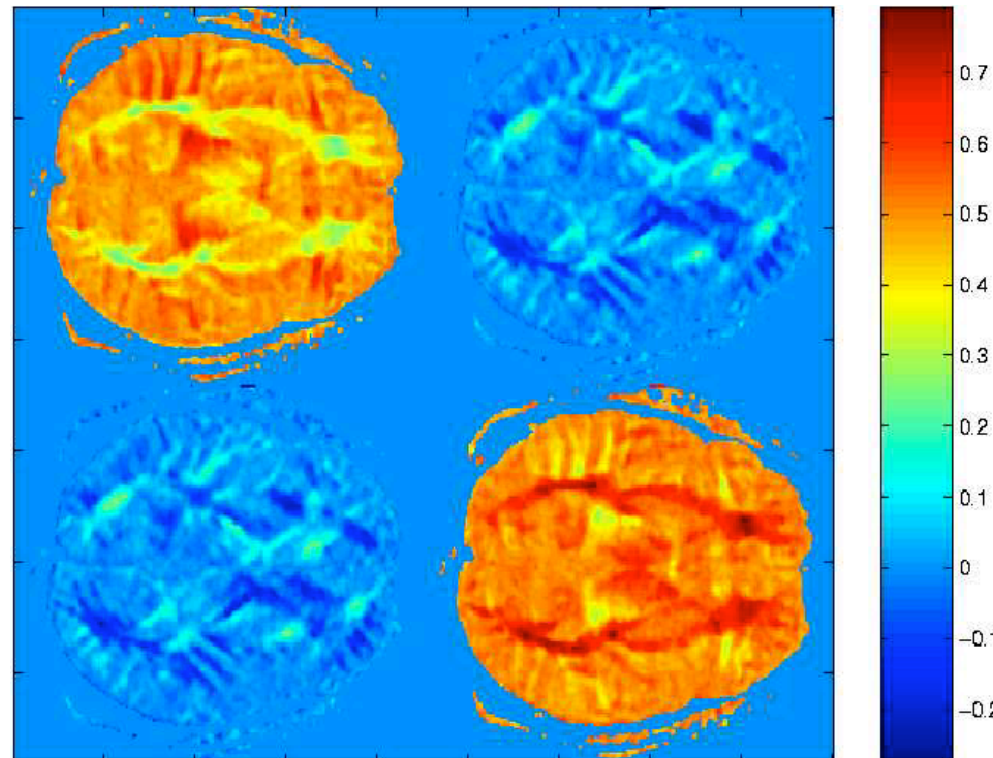


Anisotropic kernel



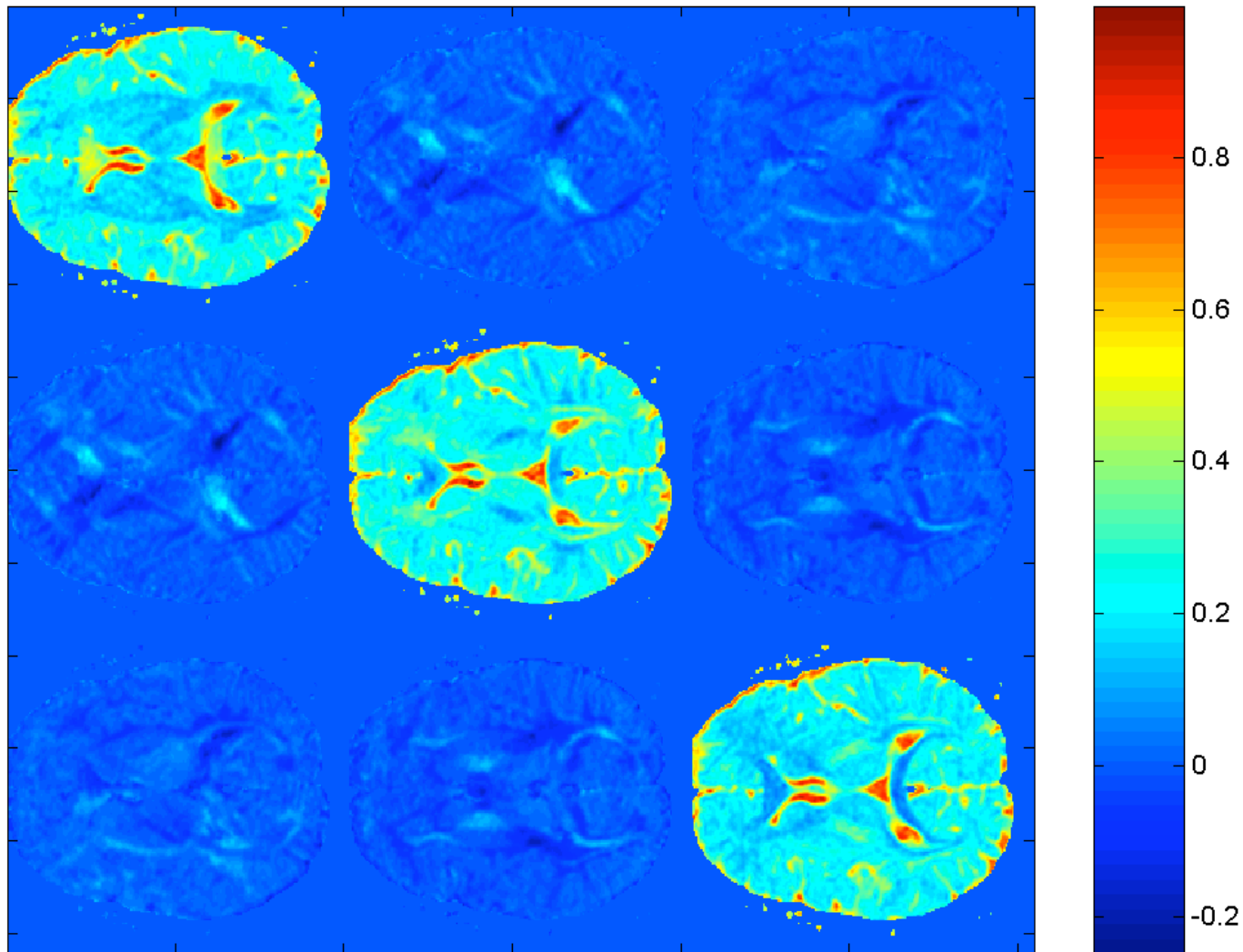
Application to DTI data

6 diffusion coefficient matrix D_{xx} , D_{xy} , D_{xz} , D_{yy} , D_{yz} , D_{zz}



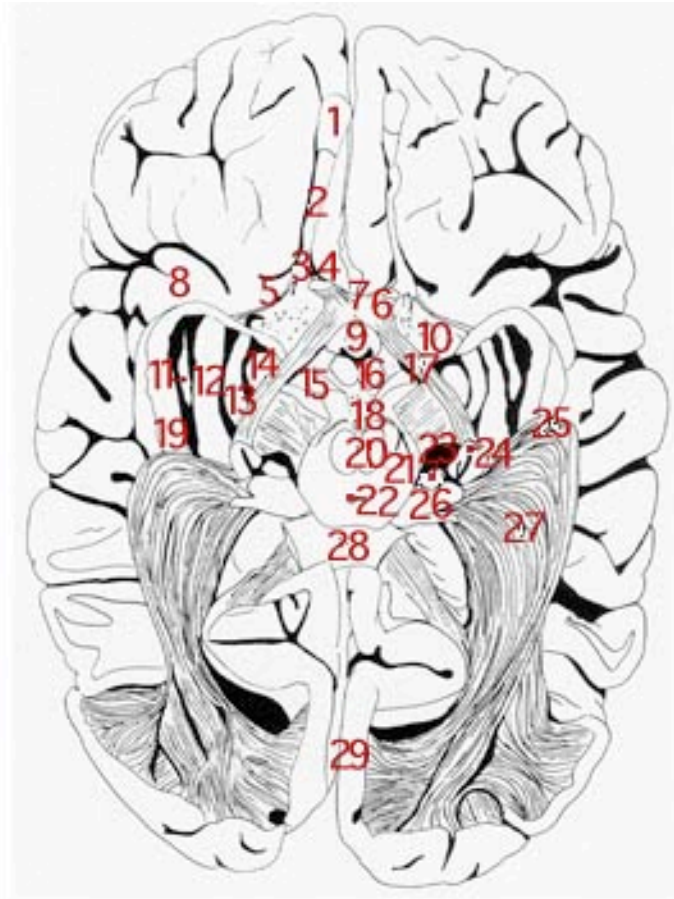
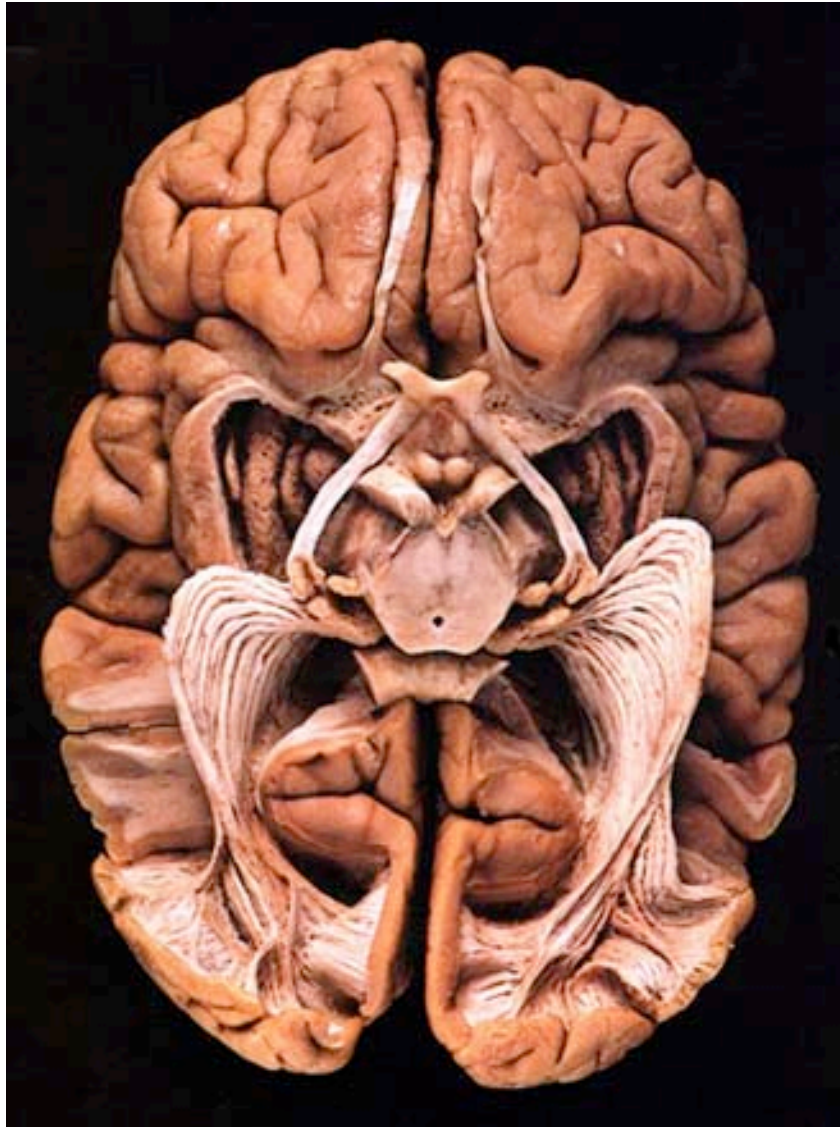
- Diffusion coefficient measures the diffusion of water molecules.
- The principal eigenvector = direction of water molecules.
- This gives indirect information about white matter fibers.

Diffusion tensor images $D = (d_{ij})$



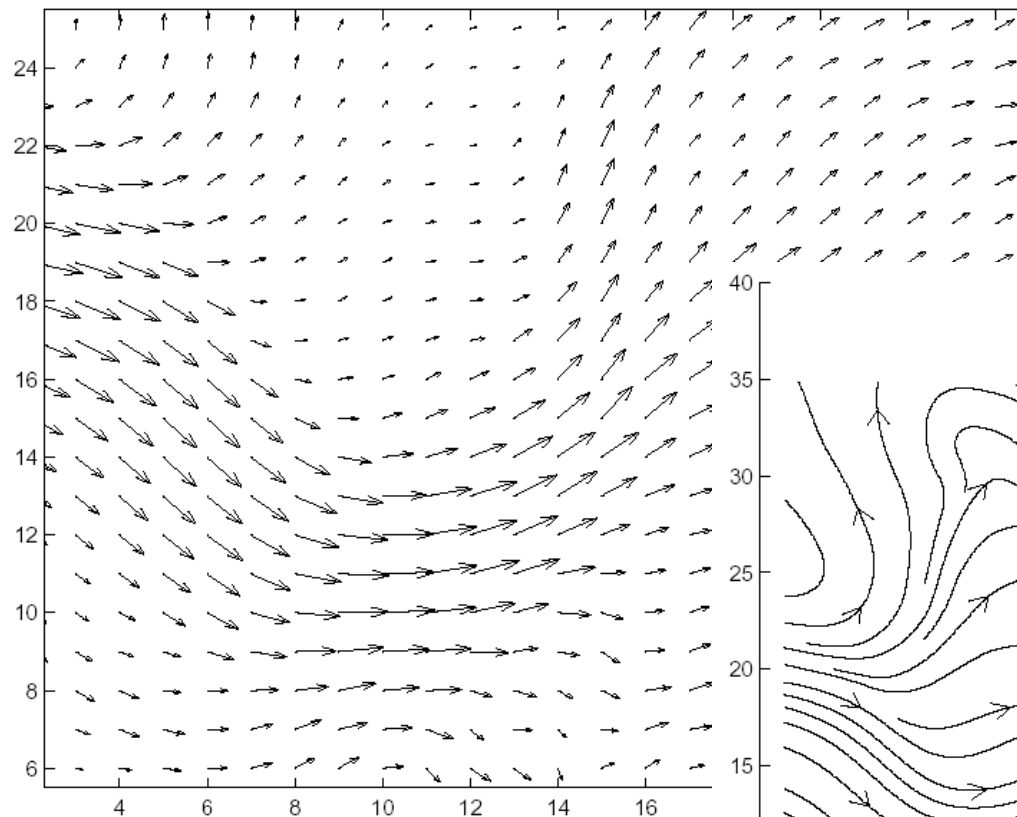
Diffusion tensors measure the diffusion of water molecules along the white fiber tracks.

White matter fibers

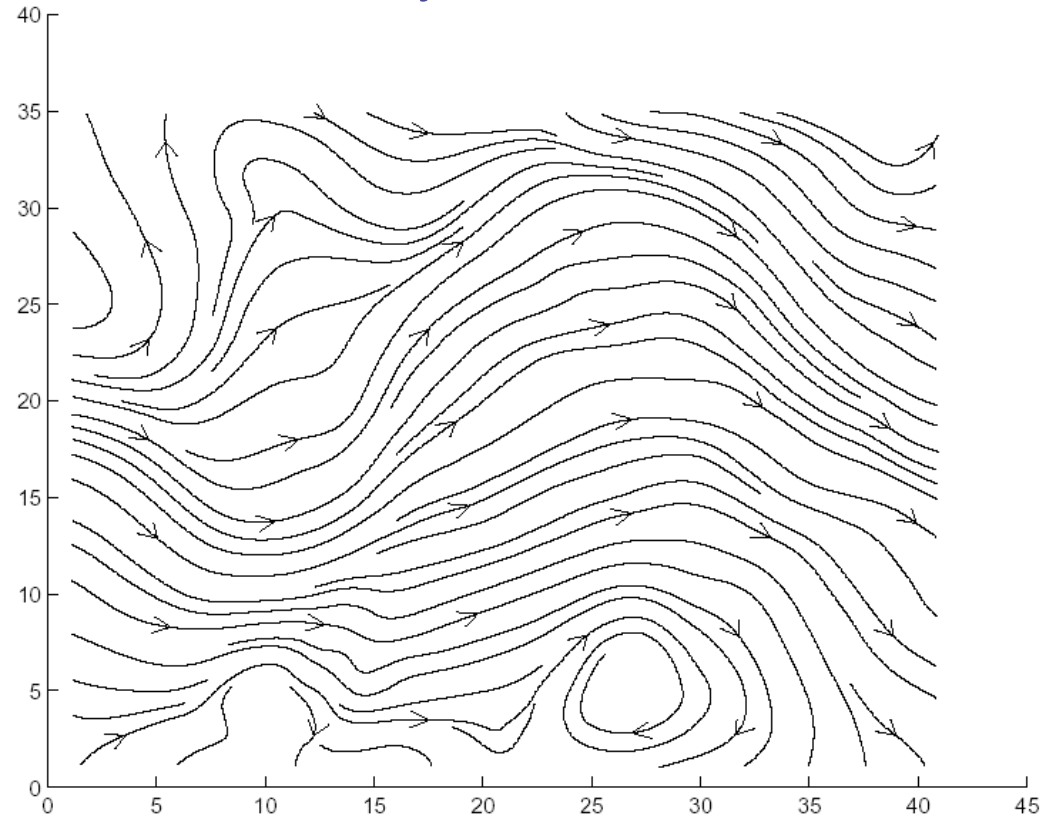


1. Olfactory bulb
2. Olfactory tract
3. Olfactory trigone
4. Medial olfactory stria
5. Lateral olfactory stria
6. Optic nerve

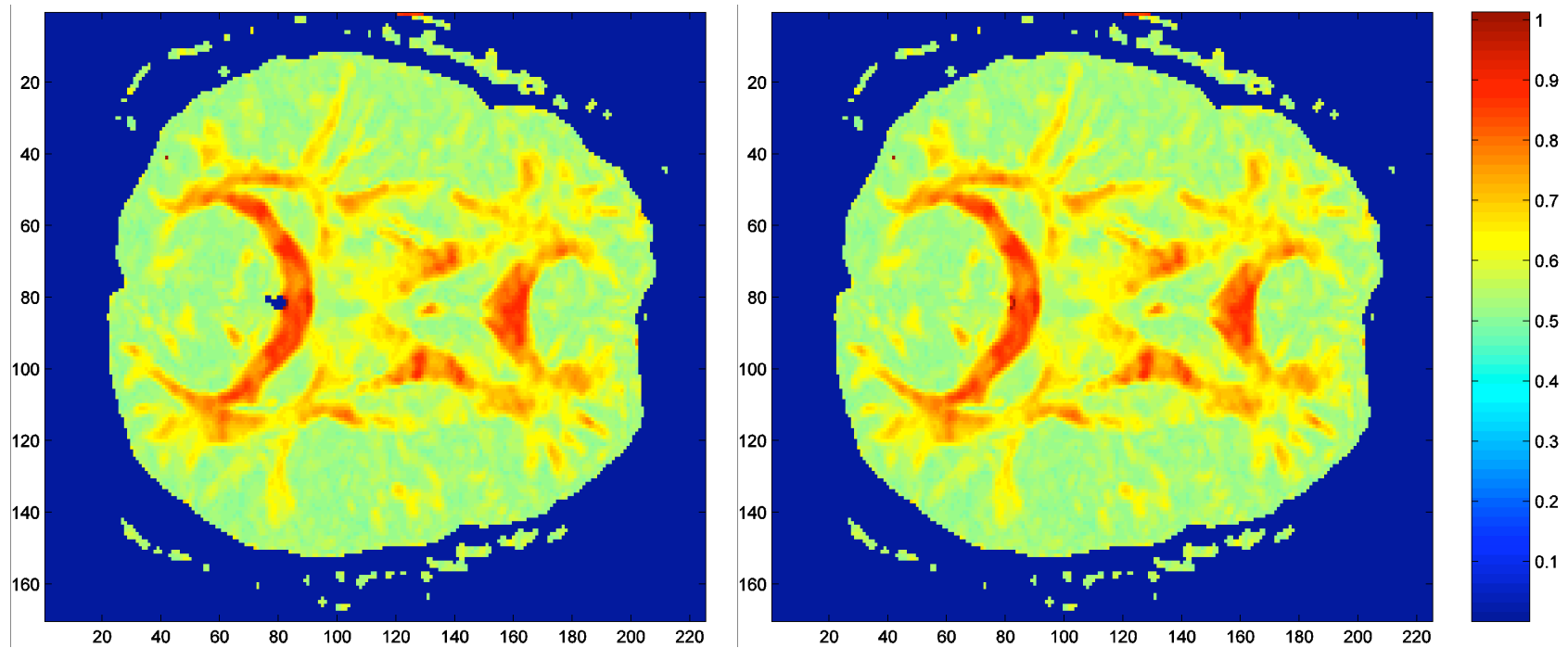
For given vector fields there exists a family of curves whose tangent is given by the vector fields.



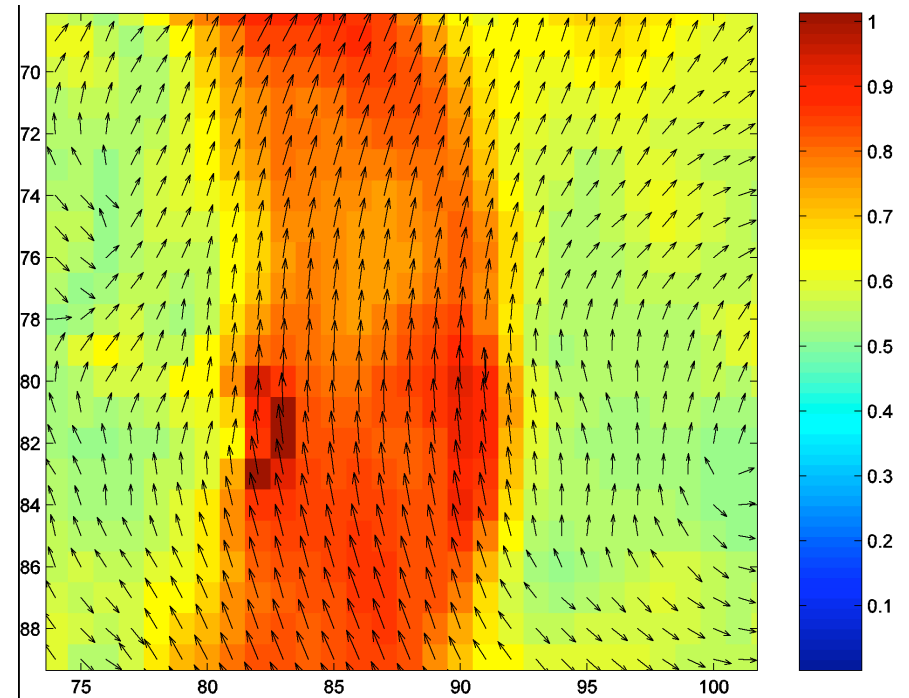
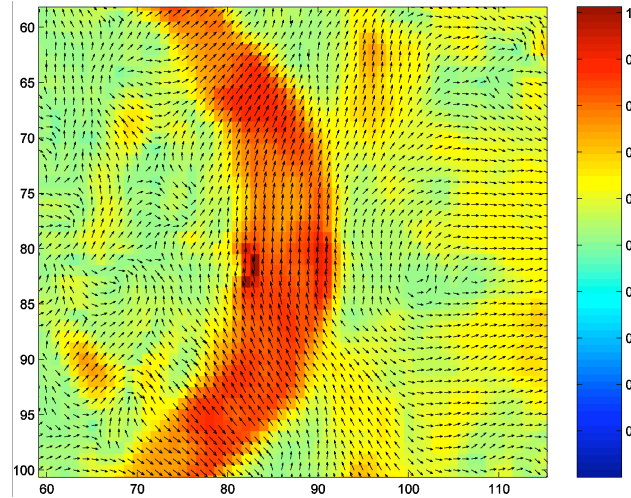
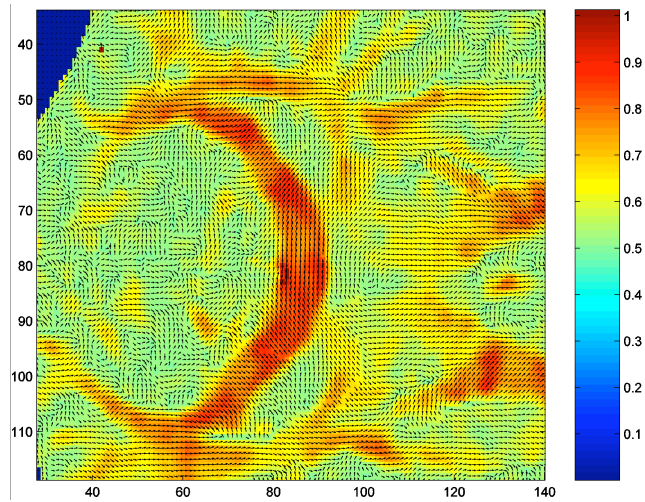
Stream lines generated
by built in MATLAB function



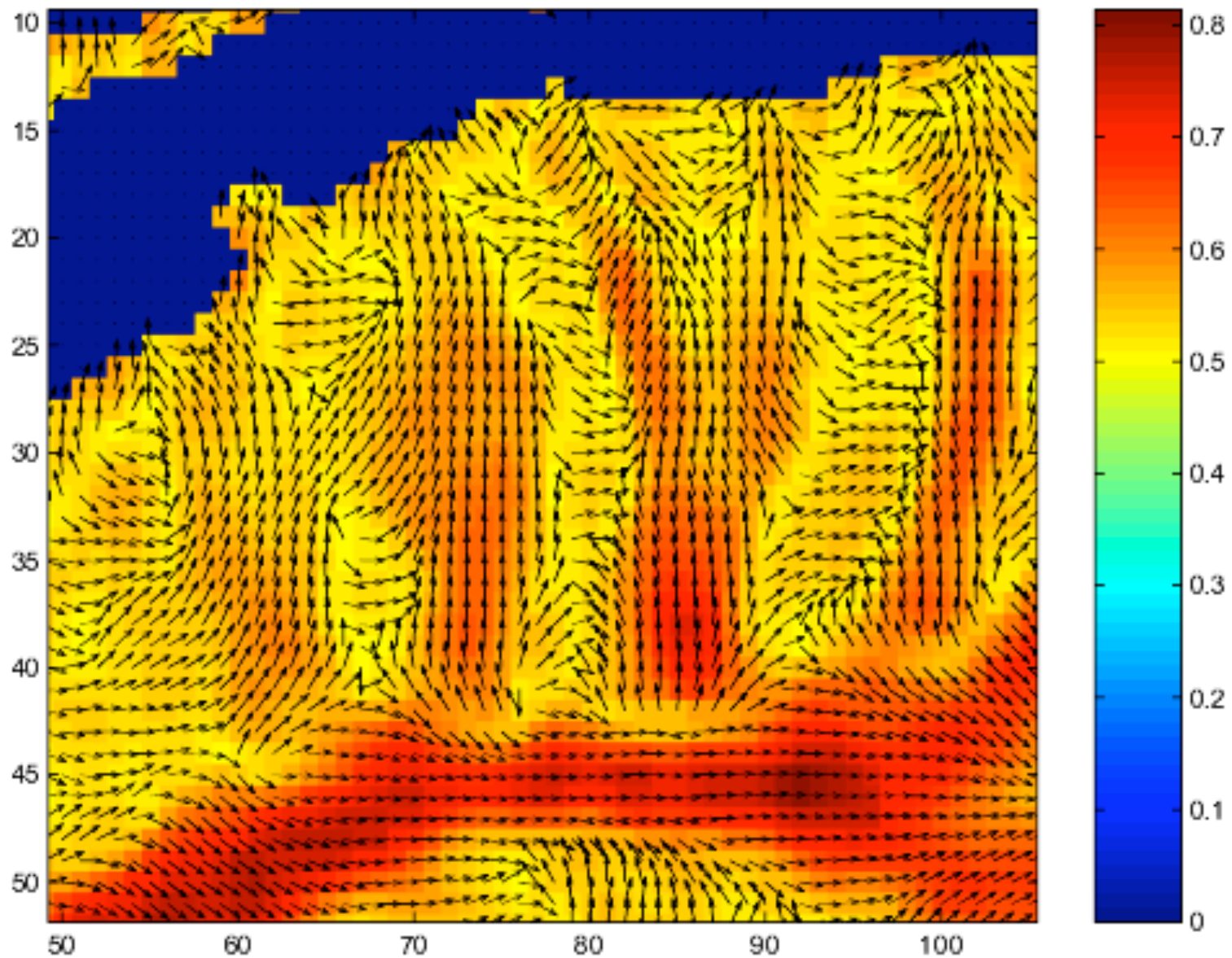
Principal eigenvalues



Principal eigenvectors



Smooth along principal eigenvectors



The stream line $\psi(t)$ corresponding to the vector field is given by

$$\frac{d\psi}{dt} = \mathbf{V}.$$

This ordinary differential equation gives a family of curves whose tangent vector is $\mathbf{V}$. The line element is

$$d\psi^2 = V_1^2 dx_1^2 + \cdots V_n^2 dx_n^2.$$

So $g_{ij} = V_i^2 \delta_{ij}$. Riemannian metric tensor

Choose $HH' = 2tG, G = (g_{ij})$. In this way we smooth more along the larger metric distance.

Relation to Diffusion equation

Let $HH' = 2tD$. Then $K_H(\mathbf{x}) = K_{\sqrt{2t}D^{1/2}}(\mathbf{x})$. Call this kernel $K_t(\mathbf{x})$

$$K_t(\mathbf{x}) = \frac{\exp(-\mathbf{x}'D^{-1}\mathbf{x}/4t)}{(4\pi t)^{n/2}(\det D)^{1/2}}.$$

$$f(\mathbf{x}, t) = K_t * g(\mathbf{x}), t \in \mathbb{R}^+$$

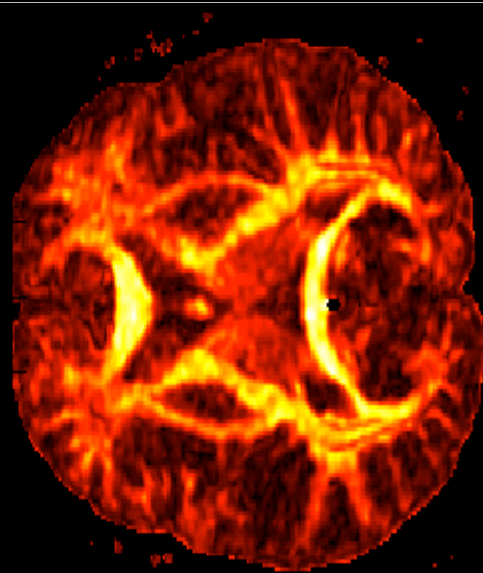
Theorem: $f(\mathbf{x}, t) = K_t * g(\mathbf{x})$ is a unique solution of

$$\frac{\partial f}{\partial t} = \nabla \cdot (D\nabla f) \tag{2}$$

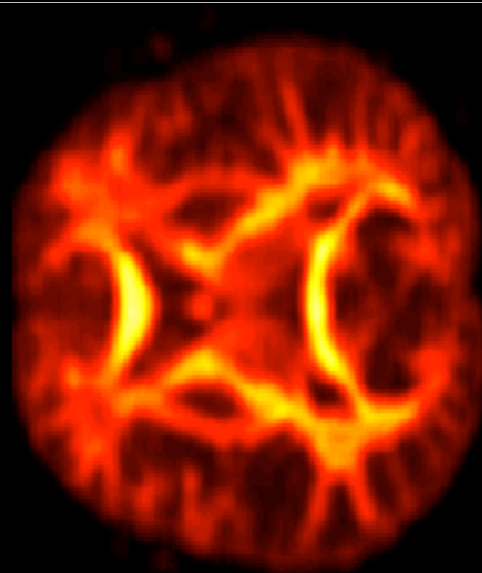
with initial condition $f(\mathbf{x}, 0) = g(\mathbf{x})$. In vector-free notation,

$$\frac{\partial f}{\partial t} = \sum_{i,j=1}^n d_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{3}$$

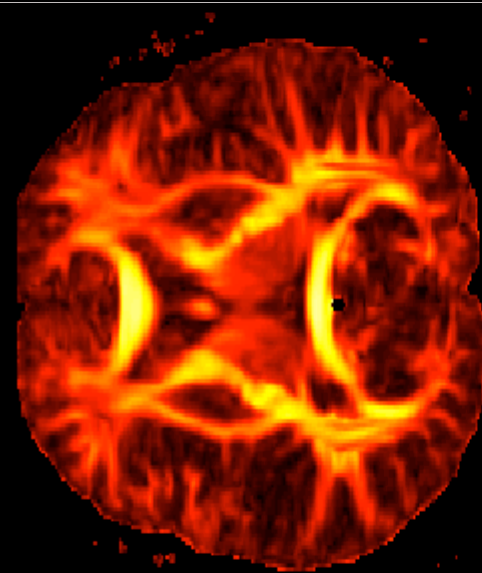
The natural Riemmanian metric tensor associated with diffusion process is $G = D$ (De Lara, Annals of Probability, 1995).



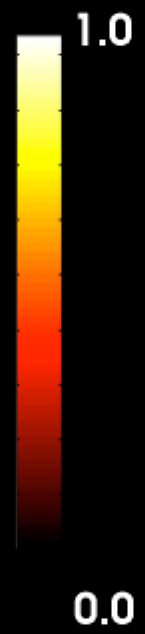
FA map



Isotropic



Anisotropic



Transition probability

Let $P_t(\mathbf{p}, \mathbf{q})$ be the *transition probability density* of a particle going from $\mathbf{p}$ to $\mathbf{q}$ under diffusion process. This is the conditional probability density of the particle hitting $\mathbf{q}$ at time t when the particle is at $\mathbf{p}$ at time 0.

Property 1. $\int_{\mathbb{R}^n} P_t(\mathbf{p}, \mathbf{x}) d\mathbf{x} = 1.$

Property 2. for small t ,

$$P_t(\mathbf{p}, \mathbf{q}) \approx K_t(\mathbf{q} - \mathbf{p}).$$

$$K_t(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2} \det^{1/2} D} \exp \left(- \frac{\mathbf{x} D^{-1} \mathbf{x}}{4t} \right)$$

Application to DTI connectivity probability

$P_t(\mathbf{p}, \mathbf{q})$ *transition probability*

Chapman-Kolmogorov equation

$$P_t(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{R}^n} P_s(\mathbf{p}, \mathbf{x}) P_{t-s}(\mathbf{x}, \mathbf{q}) d\mathbf{x}$$

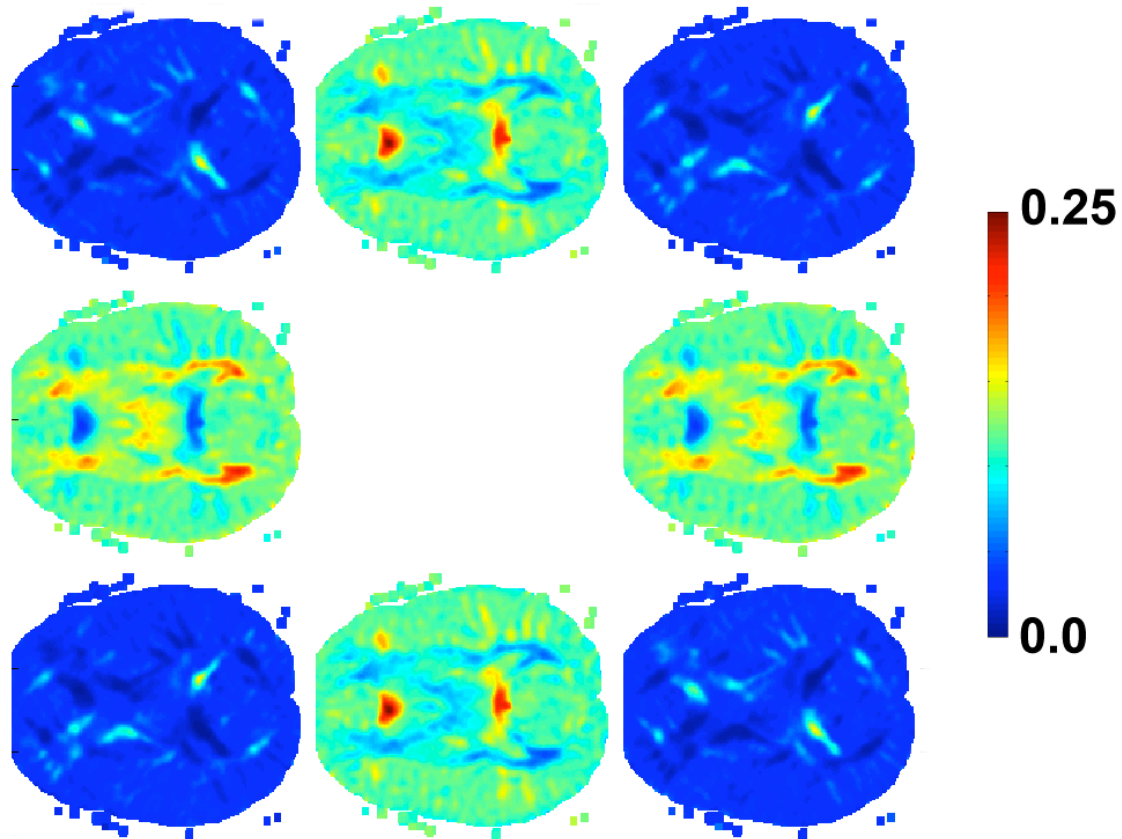
$$F_j(\mathbf{q}) = P_{j\Delta t}(\mathbf{p}, \mathbf{q})$$

$$F_j(\mathbf{q}) = \tilde{K}_{\Delta t} * F_{j-1}(\mathbf{q}) \quad F_0(\mathbf{q}) = \delta(\mathbf{p} - \mathbf{q})$$

log-transition probability

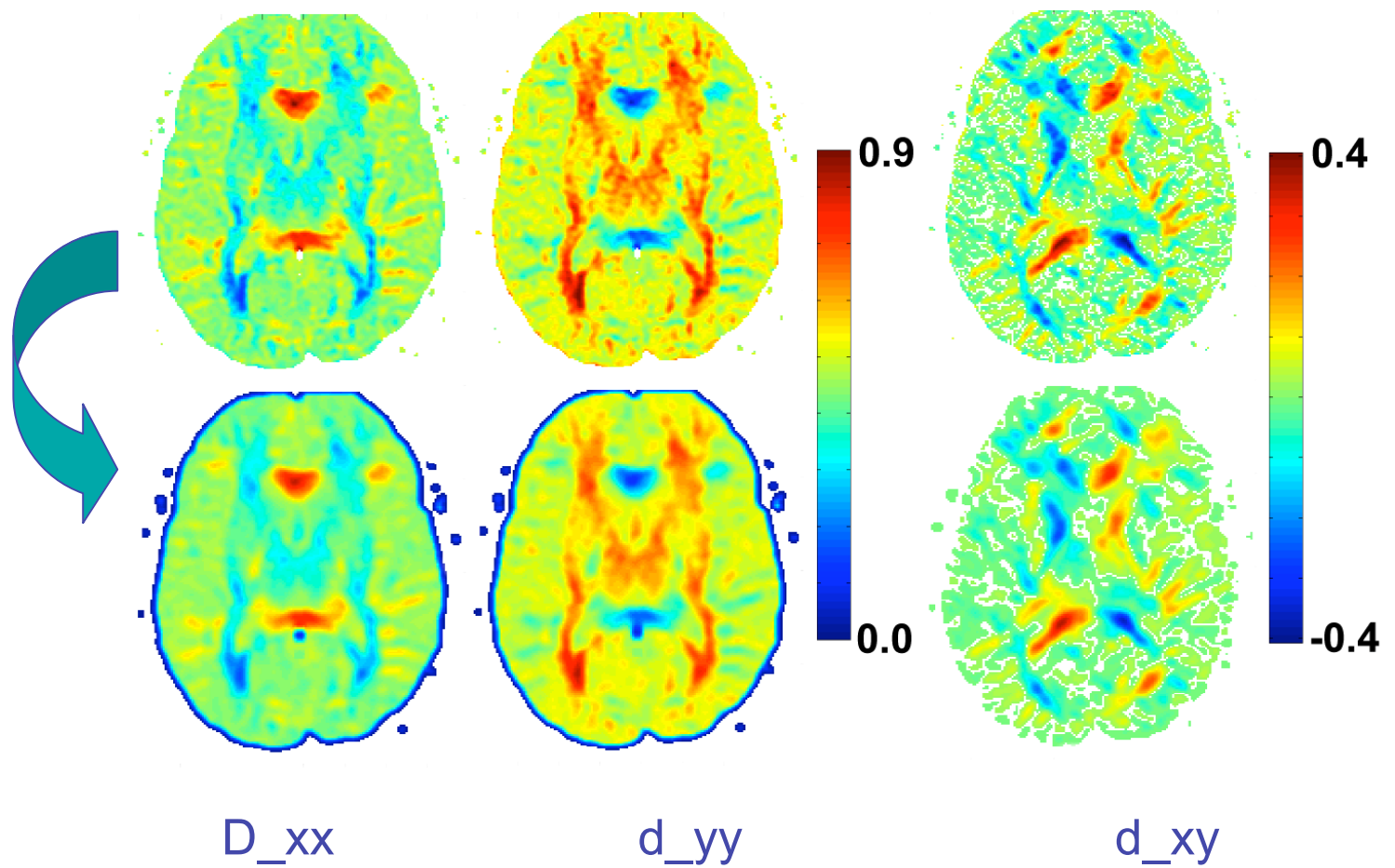
$$\rho(\mathbf{Q}) = \log \int_{\mathbf{Q}} P_t(\mathbf{0}, \mathbf{x}) d\mathbf{x}$$

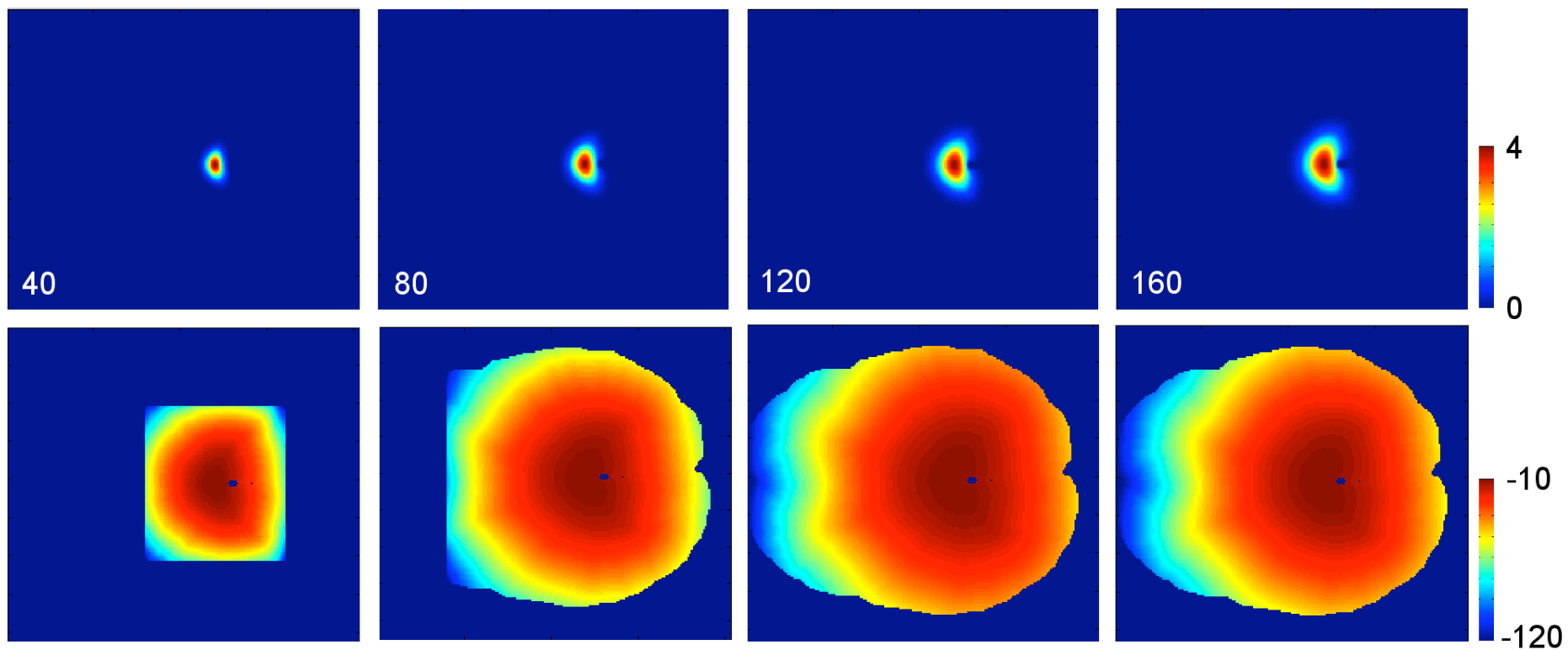
Estimated anisotropic weights



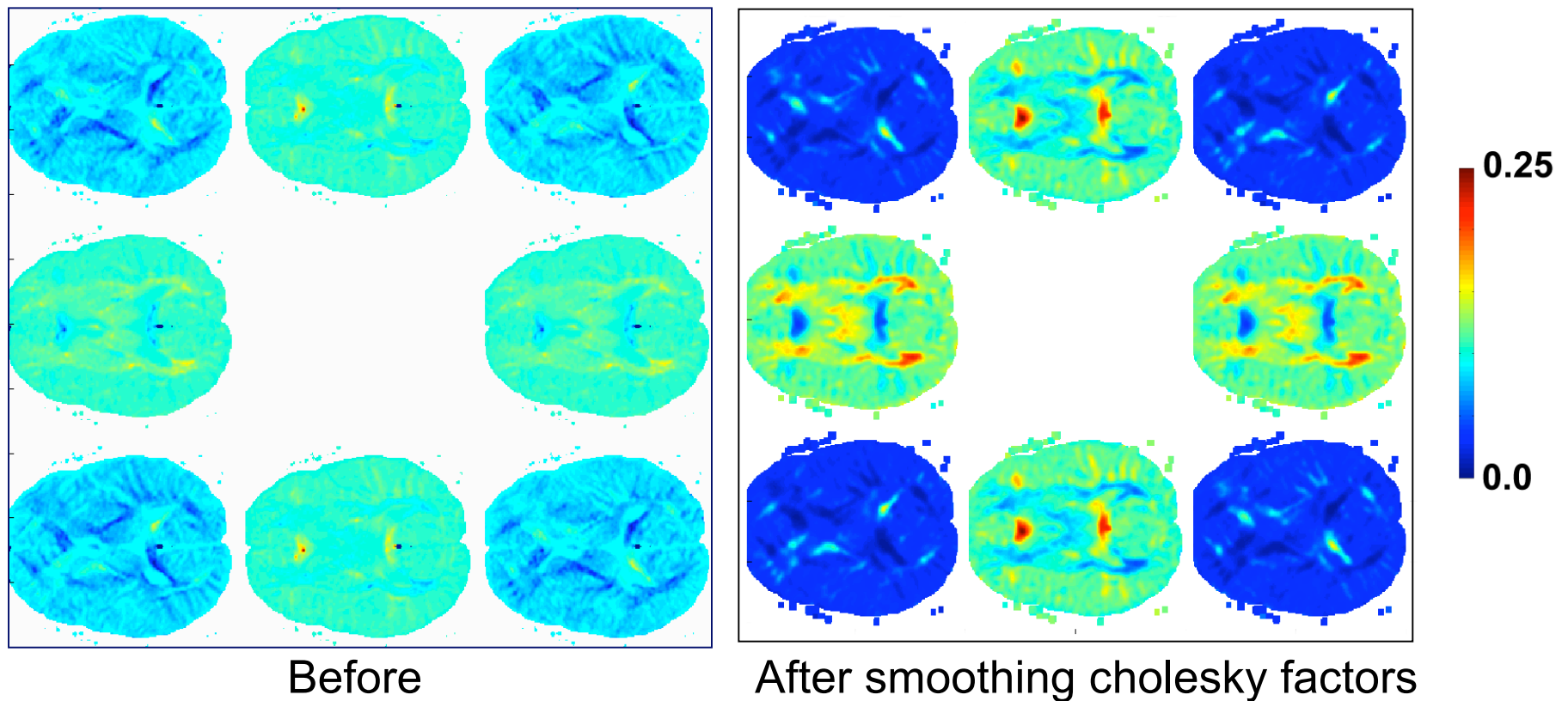
- Due to image noise, images are not smooth enough.
- Isotropic smoothing on the Cholesky factors of DTI is needed to improve the performance.

Smooth Cholesky Factors and reconstruct diffusion coefficients



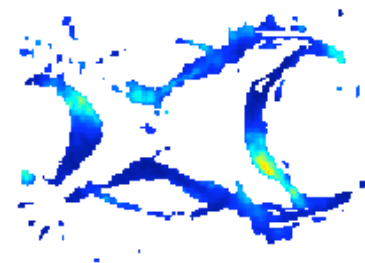
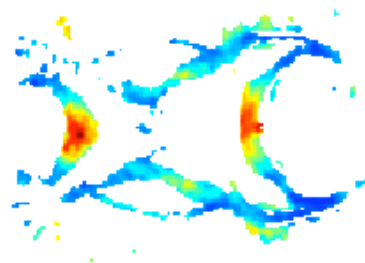
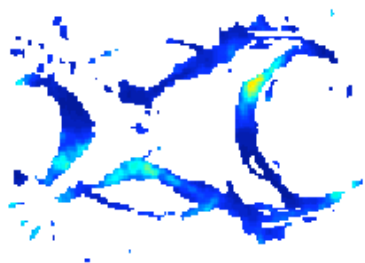
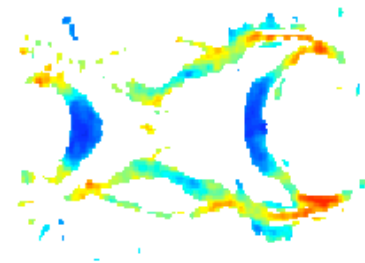
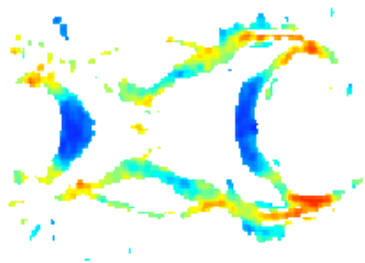
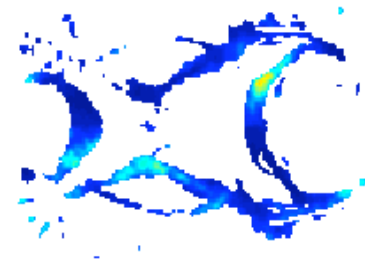
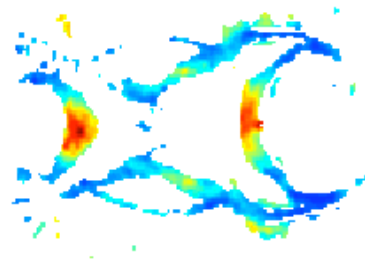
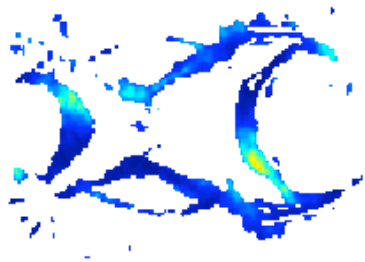


Estimated transition probability

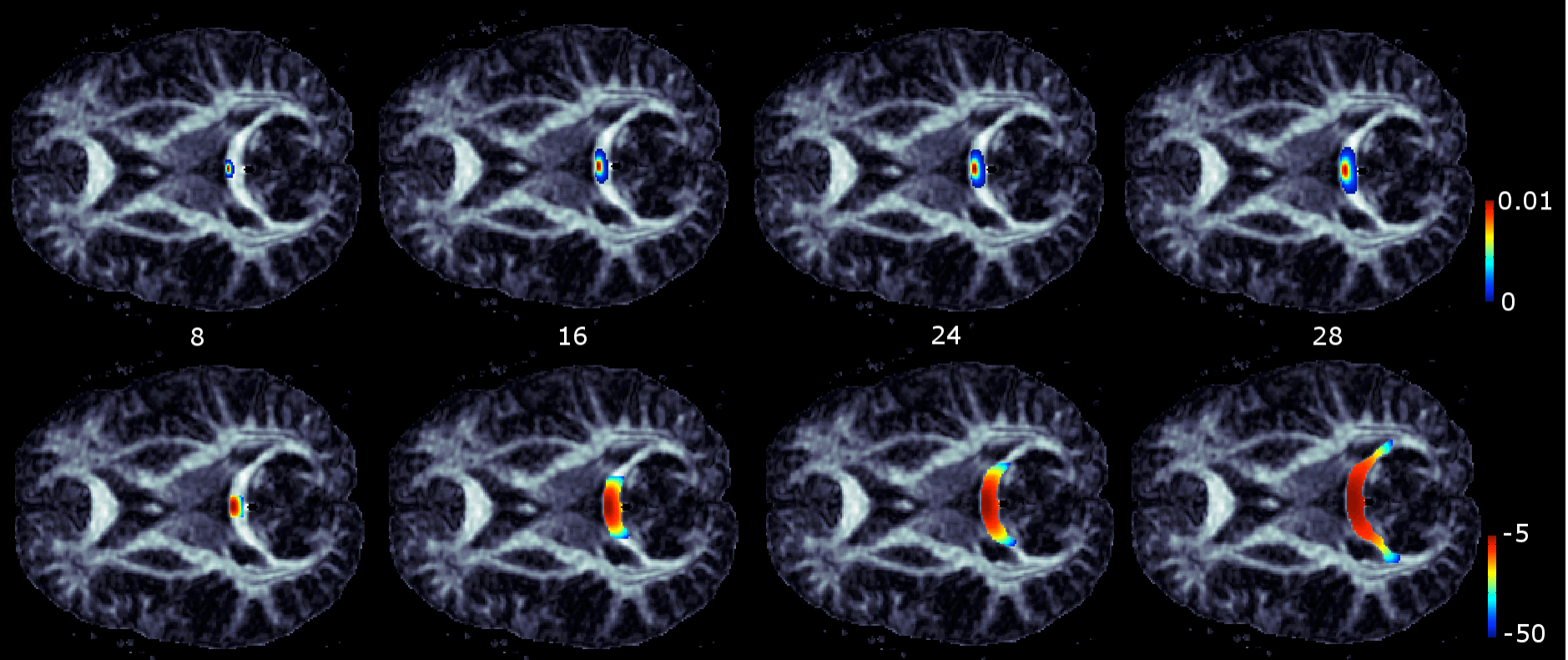


Isotropic kernel smoothing on Cholesky factors with very small bandwidth improves performance.

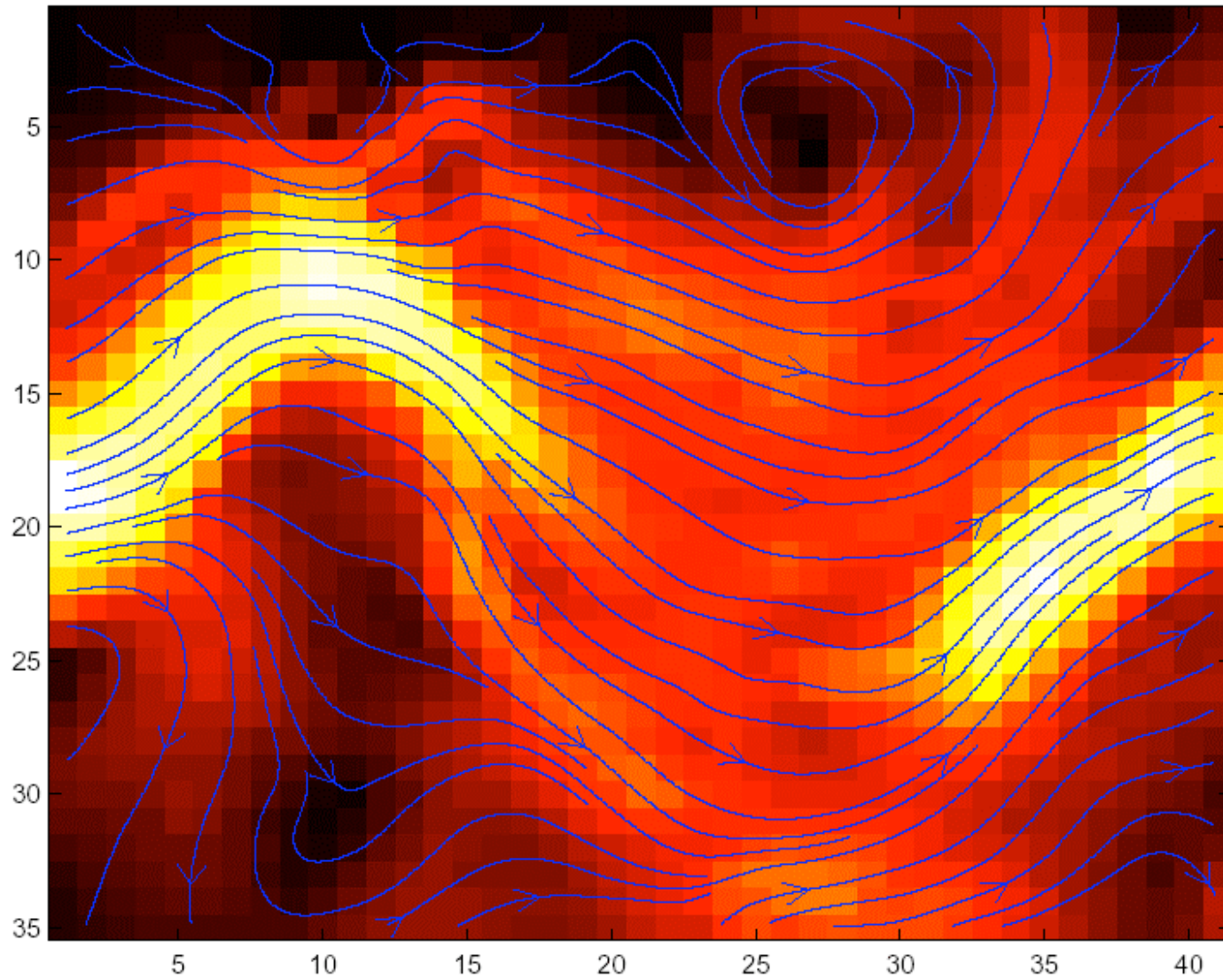
Better approach: transition probability recomputed
within $FA > 0.6$



Transition probability from a seed point in the corpus callosum.

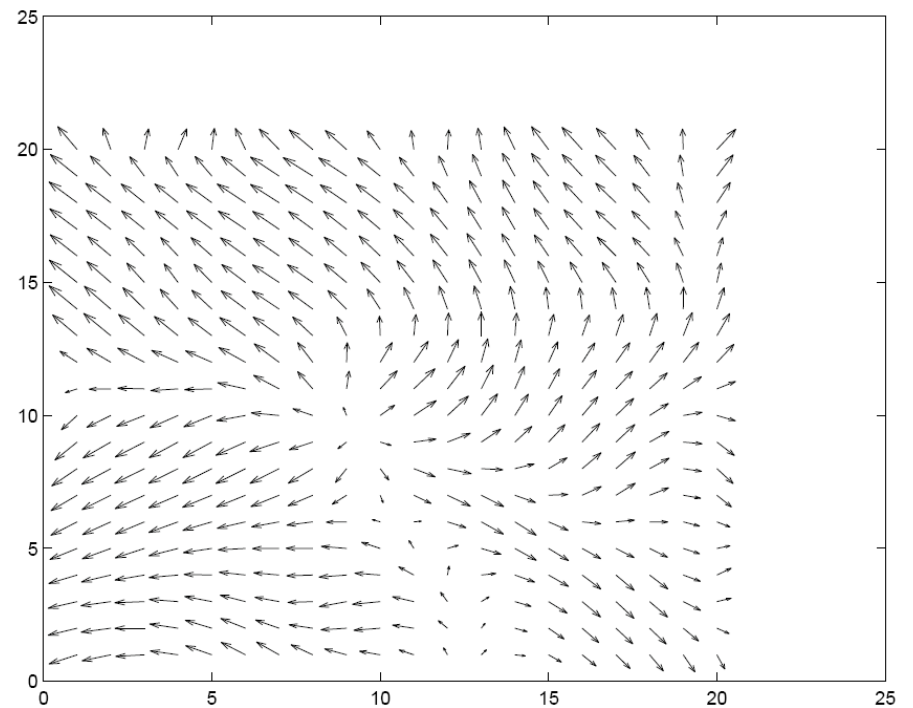
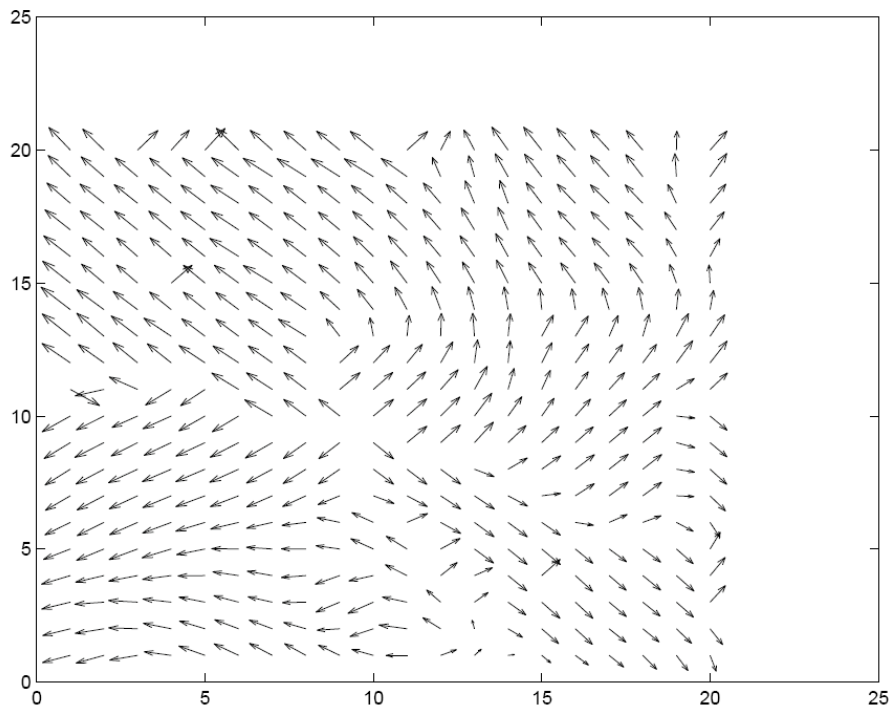


Smoothing vector fields: spline approach

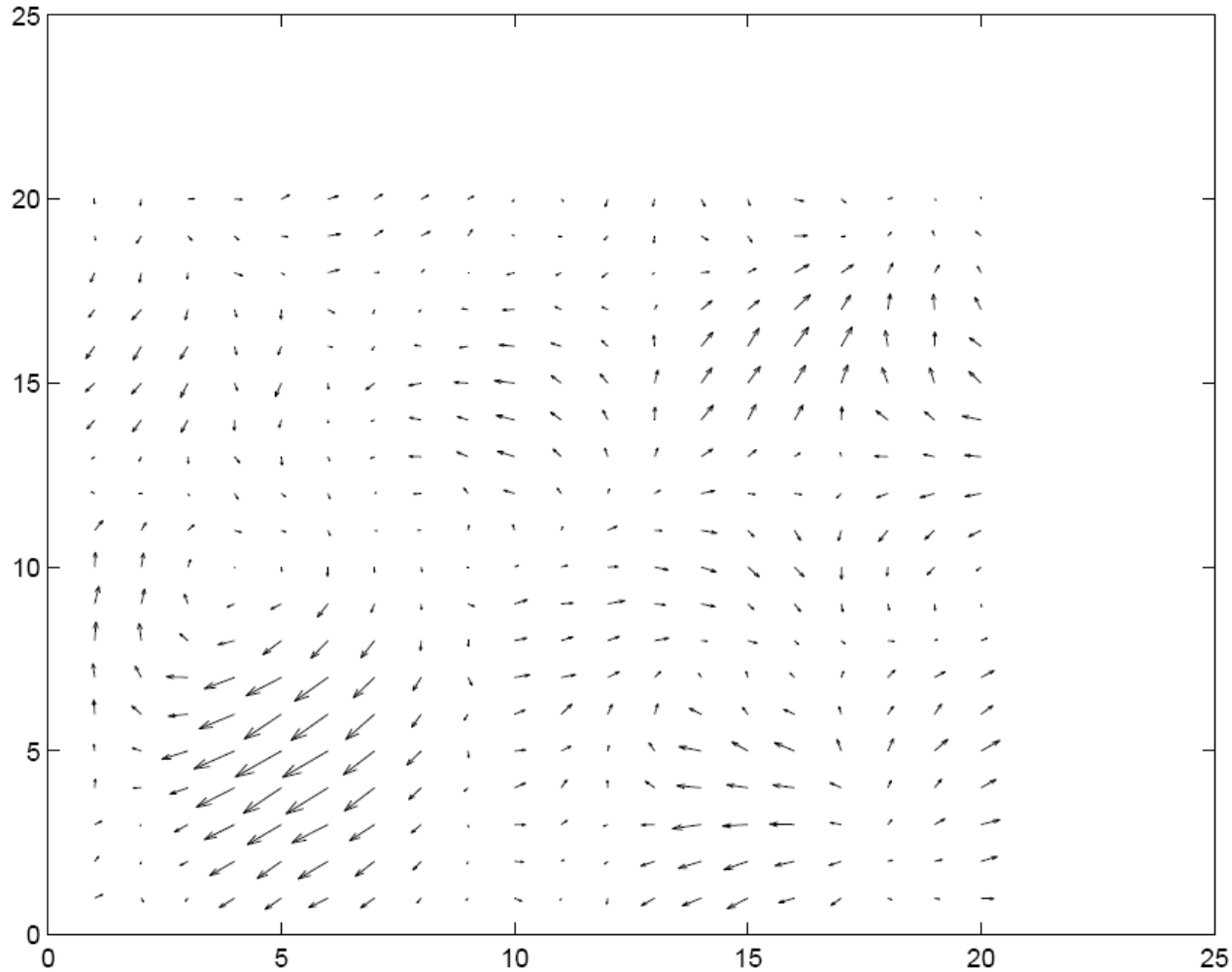


Thin-plate spline smoothing of vector fields.

This is a special case of div-curl spline.



Div-curl spline smoothing



Vector fields are decomposed into rotational (curl) and irrotational components (div).

Helmholtz theorem

The motivation for the vector spline can be seen from the decomposition of 2D vector field f . We assume that field f vanishes at infinity. f can be decomposed into divergent free (solenoidal) and rotation free (irrotational) parts using the Helmholtz decomposition:

$$f = f_{sol} + f_{irr},$$

where $f_{sol} = \nabla \times \psi$ and $f_{irr} = \nabla \phi$ for vector potential field ψ and stream function ϕ . Note that $\nabla \cdot f_{sol} = \nabla \cdot (\nabla \times \psi) = 0$ and $\nabla \times f_{irr} = \nabla \times (\nabla \phi) = 0$.

At each control point $p_j \in \mathbb{R}^d$, we have the principal eigenvector $V(p_j)$. Then we have the following stochastic model

$$V(p_j) = f(p_j) + \epsilon_j$$

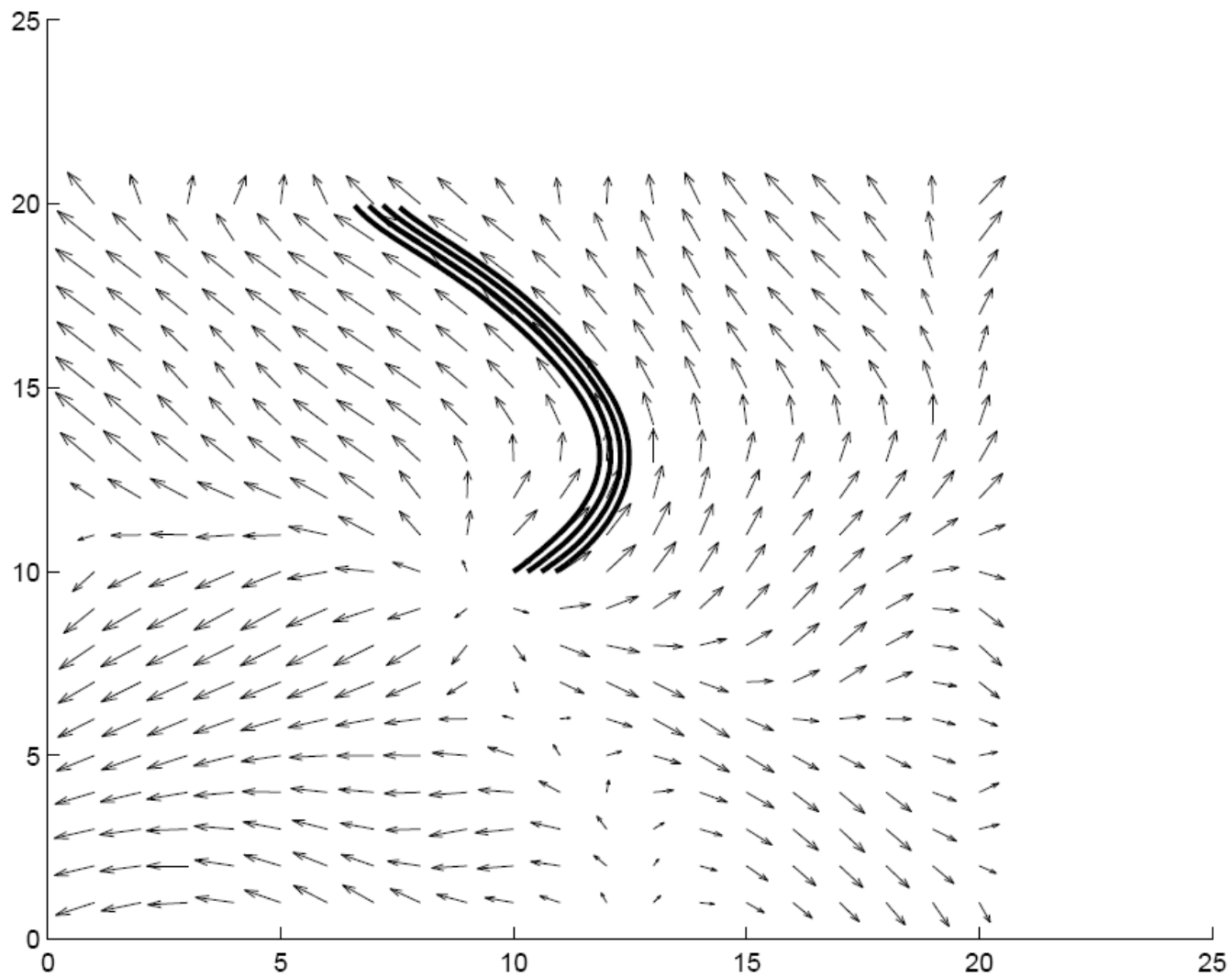
where ϵ_j is assumed to be a Gaussian white noise vector and f is the mean vector-valued function. Then we estimate the mean function f by minimizing

$$\sum_{j=1}^n \|V(p_j) - f(p_j)\|^2 + \lambda \int_{\mathbb{R}^2} \alpha \|\nabla(\nabla \cdot f)\|^2 + \beta \|\nabla(\nabla \times f)\|^2 dp.$$

This is called the second order vector spline. The first order vector spline has no gradient operator in the integral.

Div-curl spline is not efficient computationally.

Streamlines after thin-plate spline smoothing

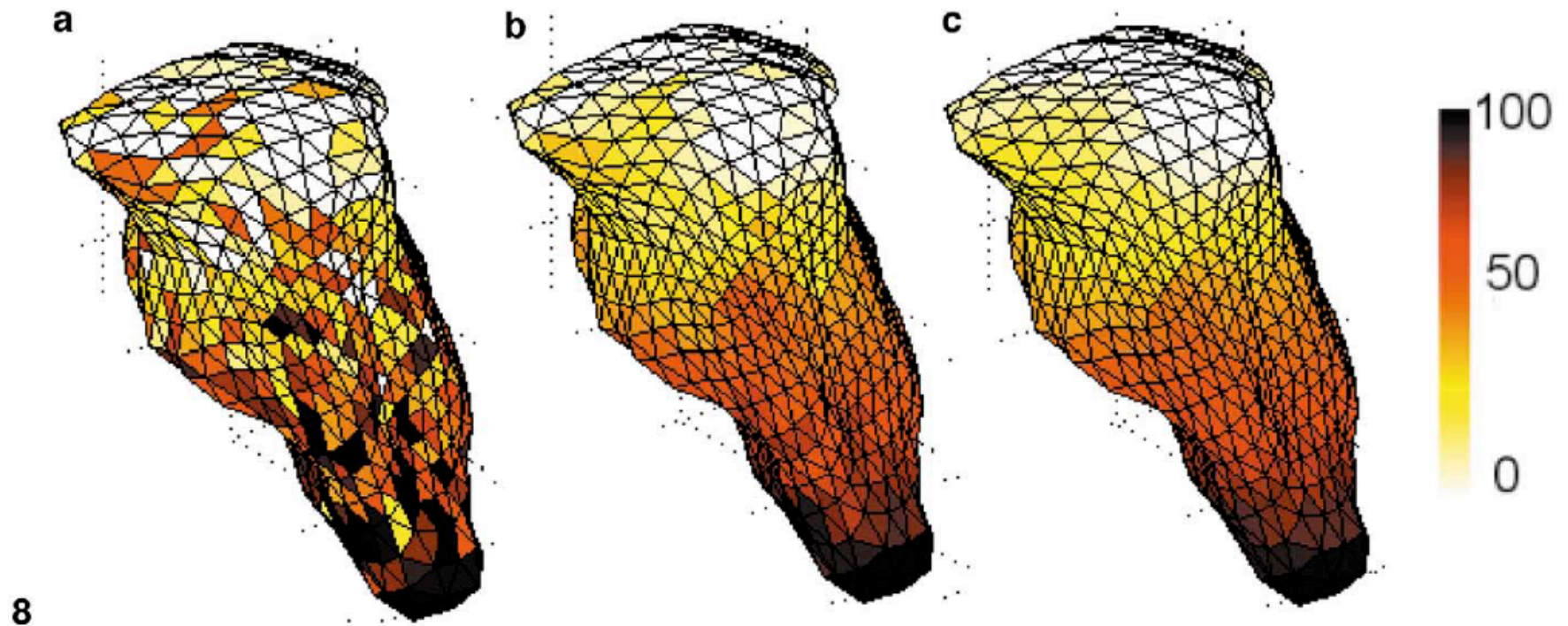


The concentration of water molecules follows the following anisotropic diffusion equation:

$$\frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C)$$

Concentration $\rightarrow$ Transition probability density

Smoothing functional data



Moo K. Chung *et al.*, Tensor-based Brain Surface Modeling and Analysis. Preceeding of Computer Vision and Pattern Recognition (CVPR) 467-473. 2003.

Smoothing data manifold



ORIGINAL IMAGE



Figure 9. Beltrami flow application on Eugenio Beltrami's portrait [Kimmel et al].

Weights for smoothing is determined by the geodesic distance between two neighboring sample points. Metric tensor approach.

Lecture 12 Topics

Guest Lecture by Prof. Li Shen

**SPHARM representation
and
Support Vector Machine**