

Medical Image Analysis

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Lecture 17.

Fractal Dimension Analysis

April 24, 2007

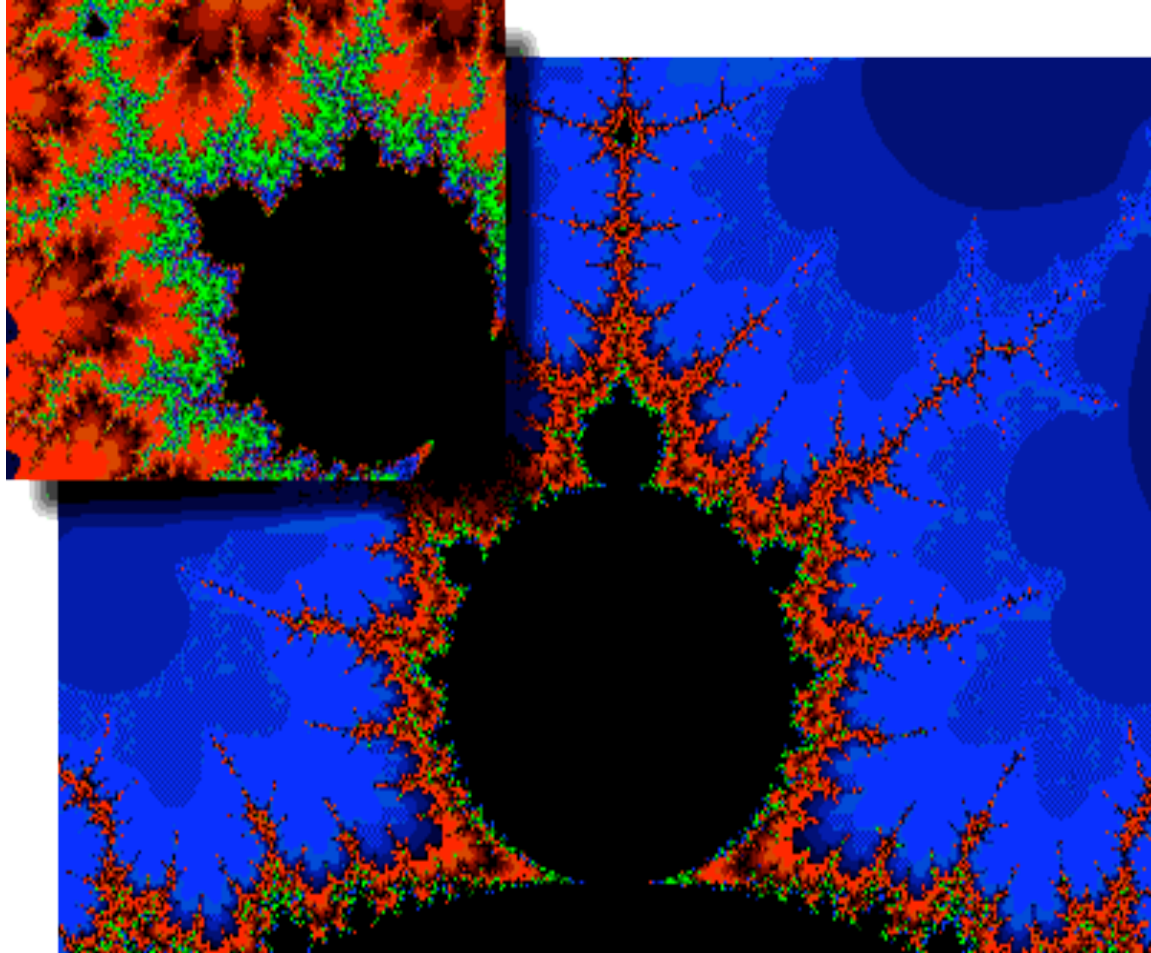
Image Complexity

- Motivation: why we are interested in studying image complexity?
- Image complexity can characterize an image and the underlying clinical statue (existence of cancer, Alzheimer's disease).
- The most widely used image complexity measure is the ***fractal dimension***.
- We will study ***fractals*** and ***fractal dimension***.

Use of Fractals

- Quantification of biological systems (most complex and chaotic in science).
- Fractal image compression: it can achieve the compression ratio of up to 600:1.
- Fractal rendering: realistic computer rendering of clouds, rocks, shadows

Examples of fractals



Fractals can be generated by recursive formula in a complex plane.

For instance, the following formula will generate the left fractal:

$$Z_{n+1} = Z_n + C$$

History of fractals

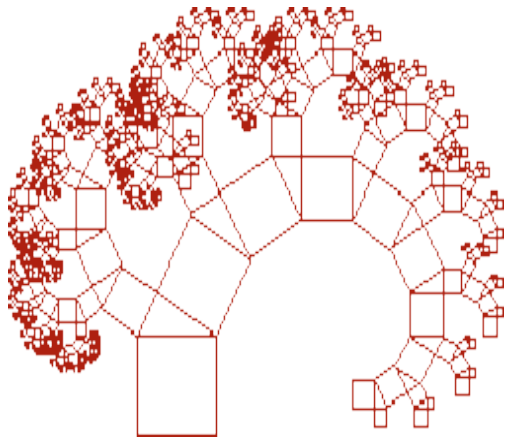


- Benoit B. Mandelbrot (1960). *Fractal geometry of nature*
- Mandelbrot termed ***fractals*** for geometric objects with self similarity.

Properties of Fractals

- Infinite detail at every point of the object
- Self (affine) similarity between parts and overall features of the object.
- Zoom into traditional shape: less detail
- Zoom into fractals: more detail

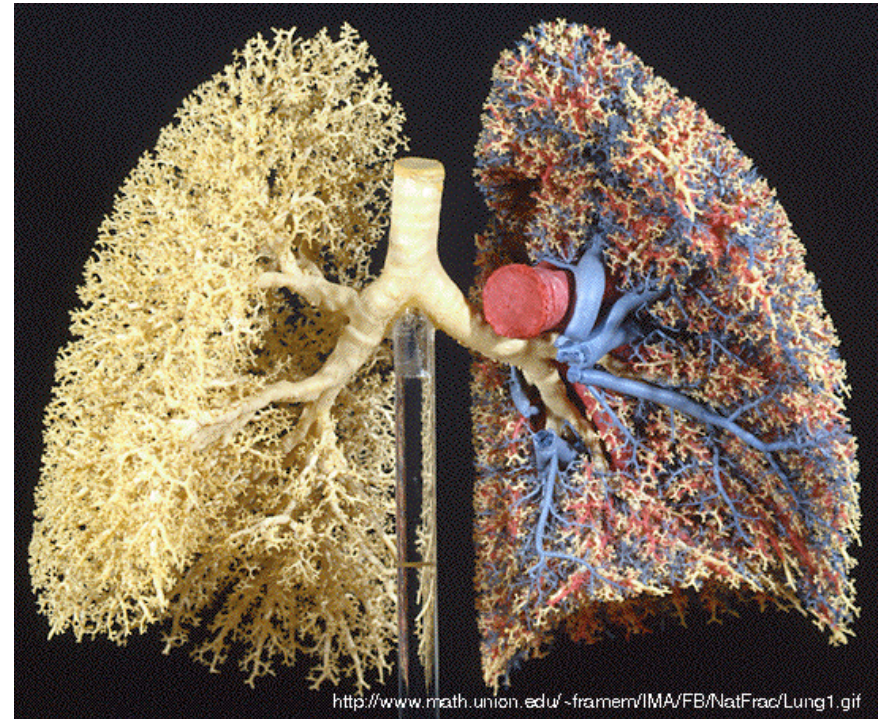
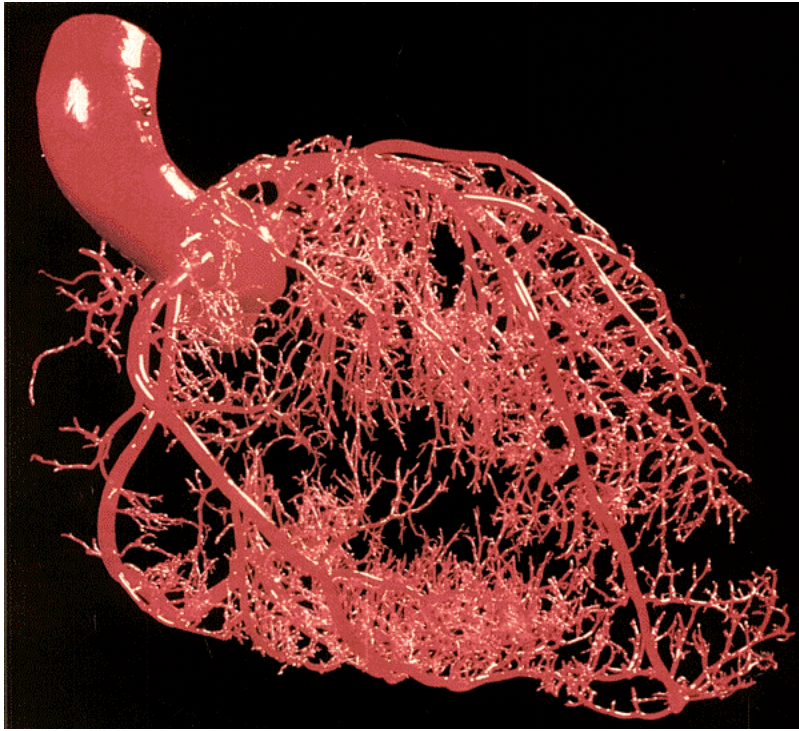
Deterministic nature of fractals



Pythagor tree

- Fractals are based on recursive mathematical formula: it can be predicted.
- Fractals are made to predict a complex and chaotic system deterministically.

Cardiovascular system



Similar to Pythagor tree

Euclidean Dimension (ED)

number of independent parameters that describes an object

- Dimension of an object measures the complexity of the object. 3D object is more complex than 1D object in general.
- Euclidean dimension: nonnegative integers 0,1,2,3,...
- Fractal dimension: 1.56, 2.49, 3.45,...

Fractal Dimension (FD)

- amount of variations in the object
- Measure of roughness/fragmentation of the object

Small FD = less jagged/complex

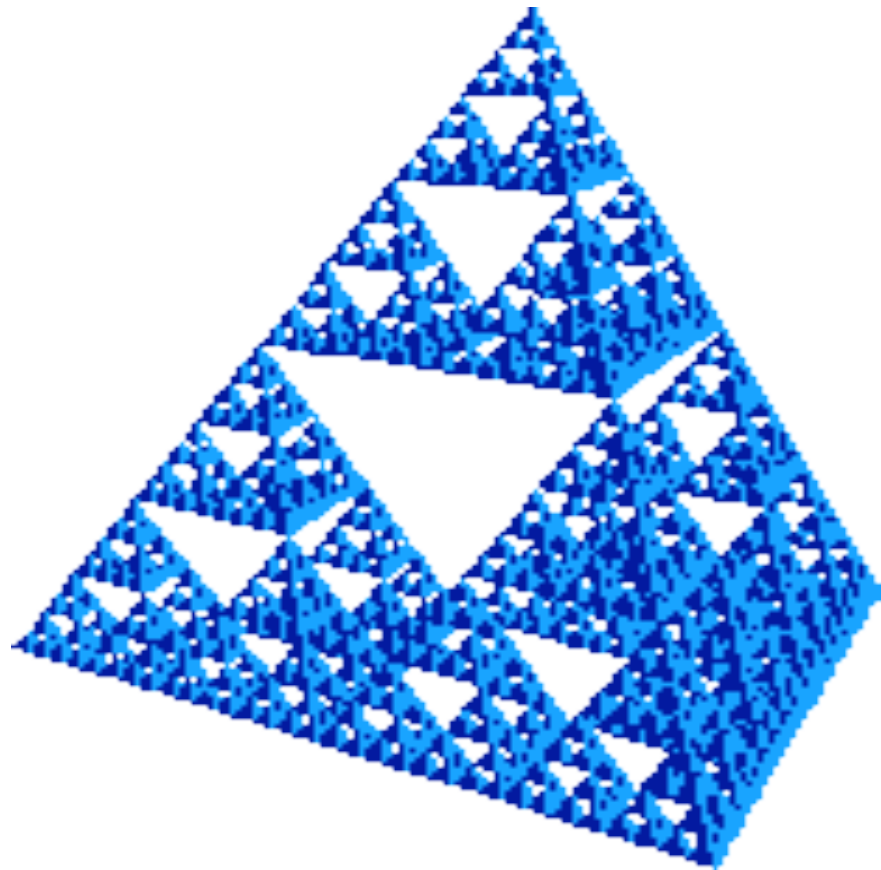
Large FD = more jagged/complex

Constructing Fractals

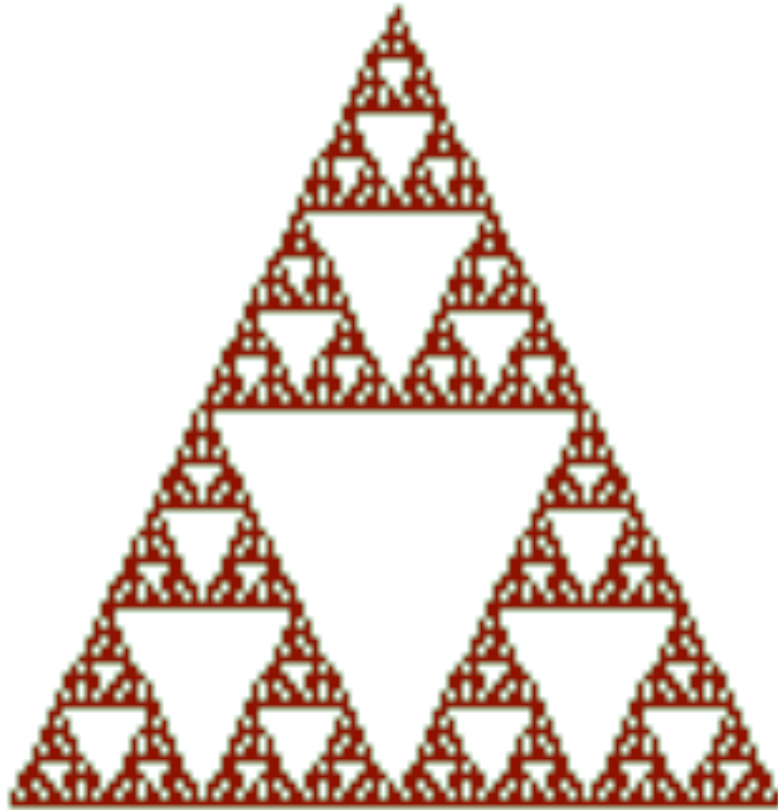
- Deterministic fractals are constructed by an iterative process with the initial configuration

Sierpinski lattice is constructed from a large triangle by recursively cutting smaller lattice.

$$FD = \ln 3 / \ln 2 = 1.58496 \dots$$

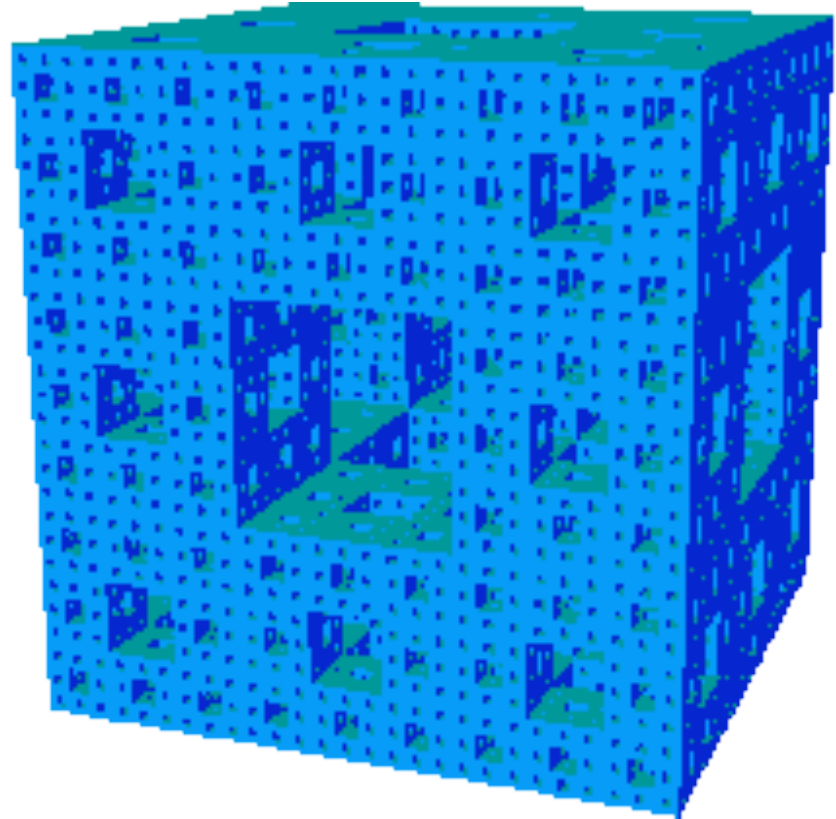


Sierpinskij Triangle

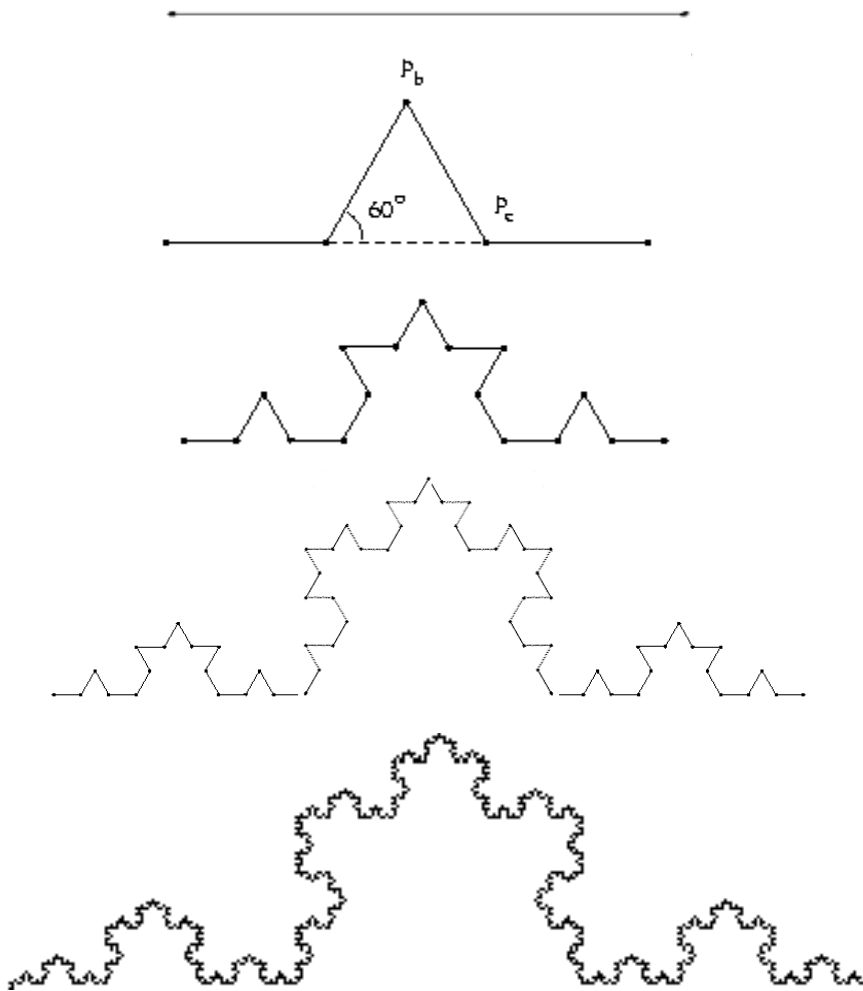


$$FD = \ln 9 / \ln 3 = 2$$

Sierpinskij Sponge

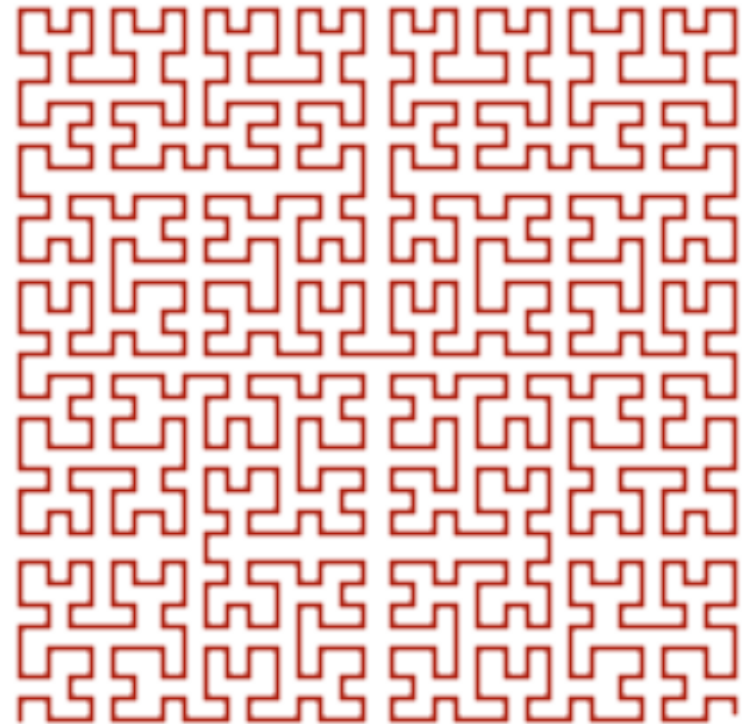


Koch curve



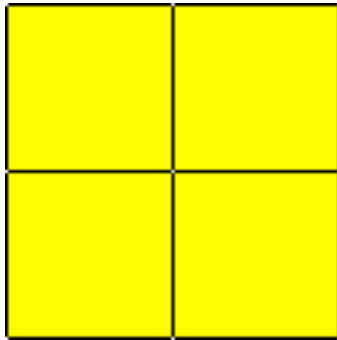
$$FD = \ln 4 / \ln 3 = 1.261859....$$

Hilbert curve

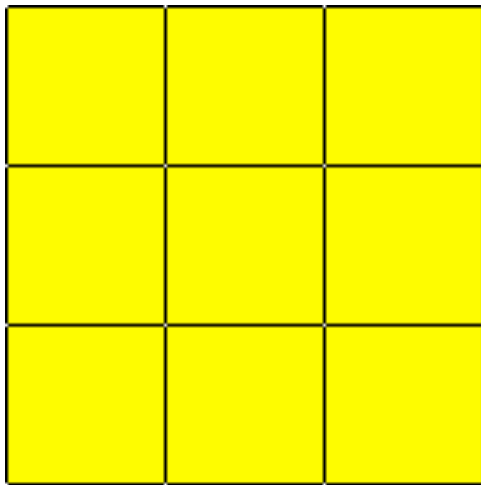


$$FD = 2$$

Example of self-similarity



Dilate by a factor of $k=2$
 $N=4$ copies of the self
similar original square.



Dilate by a factor of $k=3$
 $N=9$ copies of the self
similar original square.

K =scale factor
 N =number of copies.

Measure of self-similarity

- For the square, we have

$$k^2 = N$$

- Alternately,

$$\log_k N = 2$$



Euclidean dimension

1D example

Line segment

Original



Dilated



$k = \text{scale factor} = 2$

$N = \text{number of copies} = 2$

$$\log_k N = 1$$

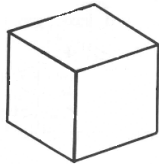


Euclidean dimension

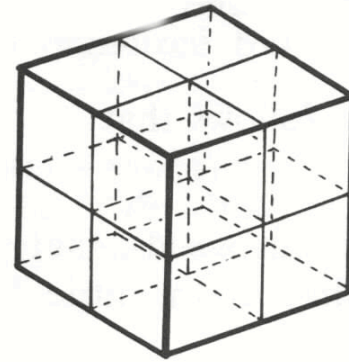
3D example

Cube

Original



Dilated



k = scale factor = 2

N = number of copies of original = 8

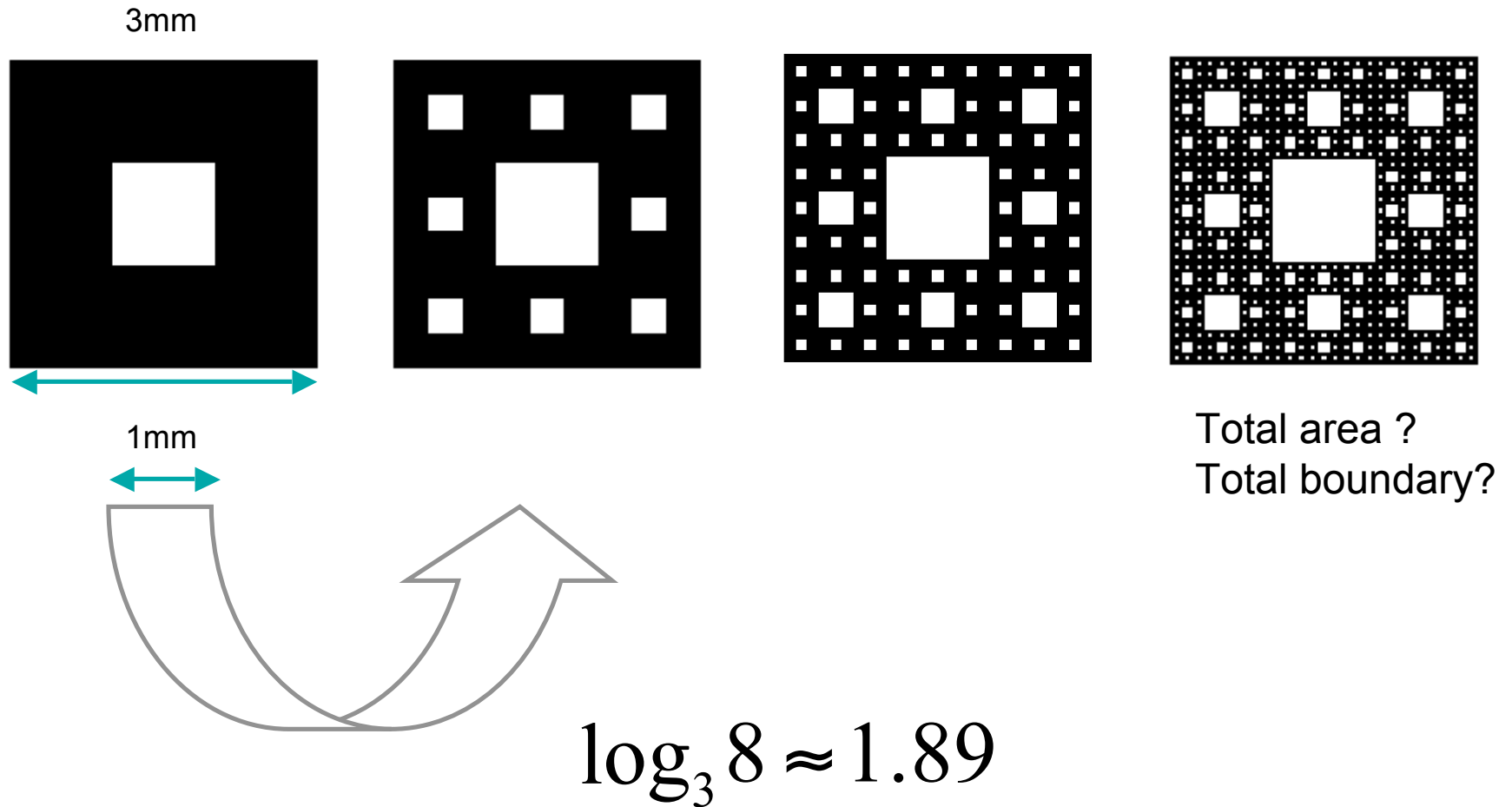
$$\log_k N = 3$$

Euclidean dimension

ED to FD

- $\log_k N$ measures the dimension of the object.
- This is the definition of the *dimension of a self-similar object*.
- We call this dimension as the **fractal dimension**.

FD computation for Sierpinski carpet



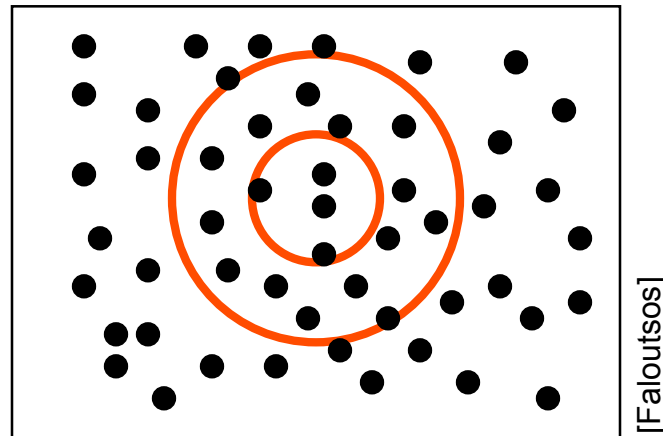
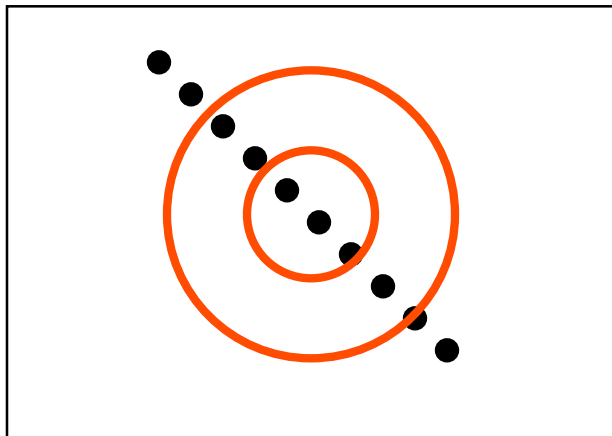
Estimating FD

- FD can be explicitly computed on fractals that are mathematical given.
- In practice, the object of interest may not be a fractal but we can still estimate FD.
- Box-counting dimension (mandelbrot). The most widely used method
- Perimeter-area dimension (Hausdorff)

Dimension of data

- FD can be taken as the approximation to the intrinsic dimension (D_i) of data, which can be different from the embedding dimension (D_e).

$$\text{count} \left(x, x' : \|x - x'\| < k \right) \approx k^{FD}$$



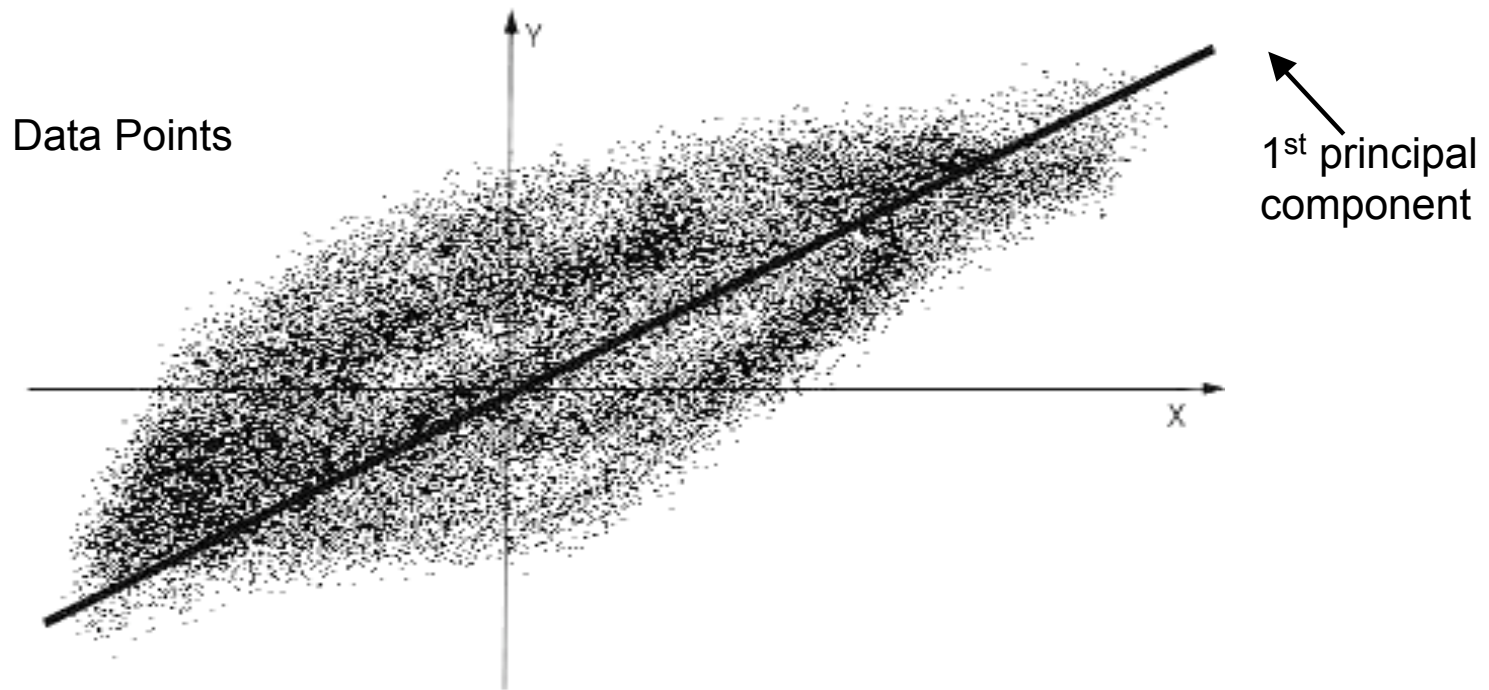
■ line in a plane:

- $D_e = 2$
- $D_i = 1$
- $FD \approx 1$

■ uniform dist. in a plane:

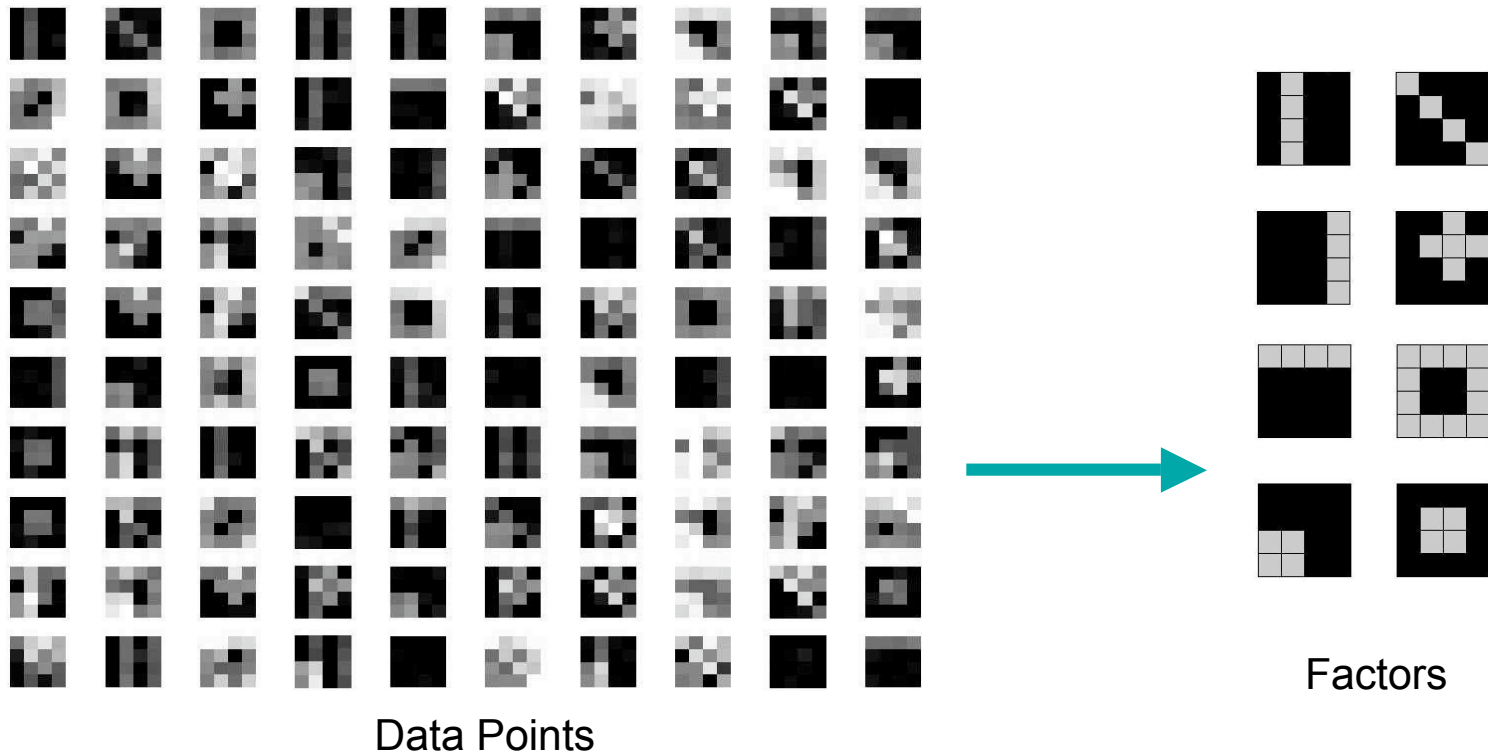
- $D_e = 2$
- $D_i = 2$
- $FD \approx 2$

FD and PCA



FD and the number of significant principal component ?

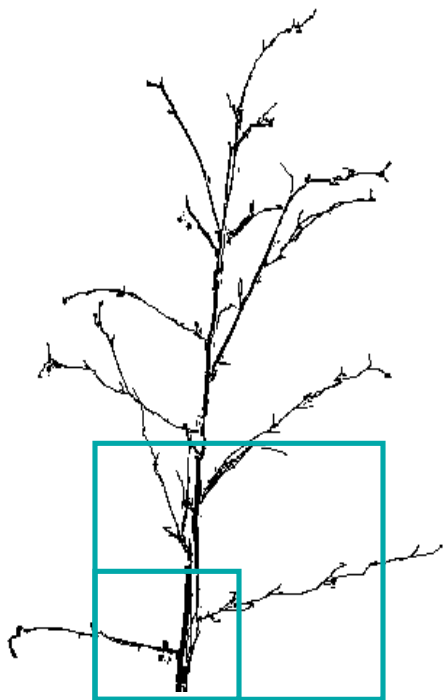
FD and Factor Analysis



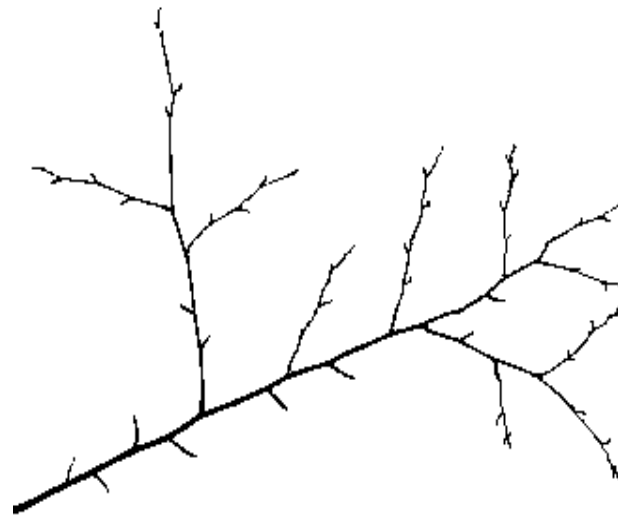
FD and the number of significant factors ?

FD and evolution

- Tree branching is related to evolution. There is a direct relationship between branching patterns and common ancestry (Bickel, 2000)



Domestic Birch



Wild Birch

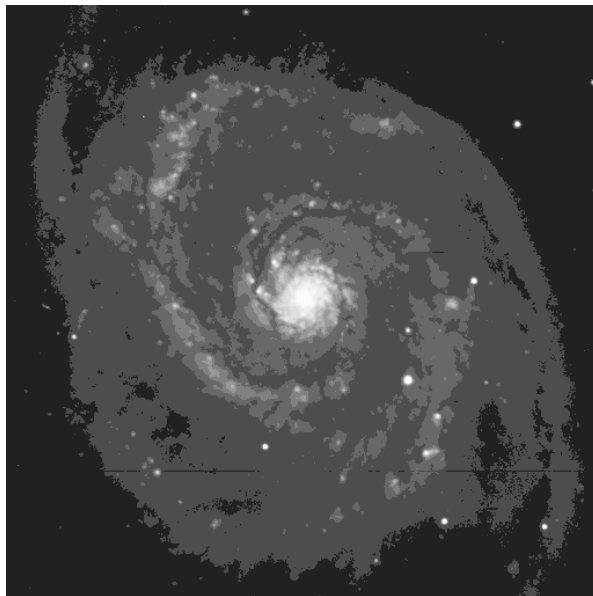


Maple

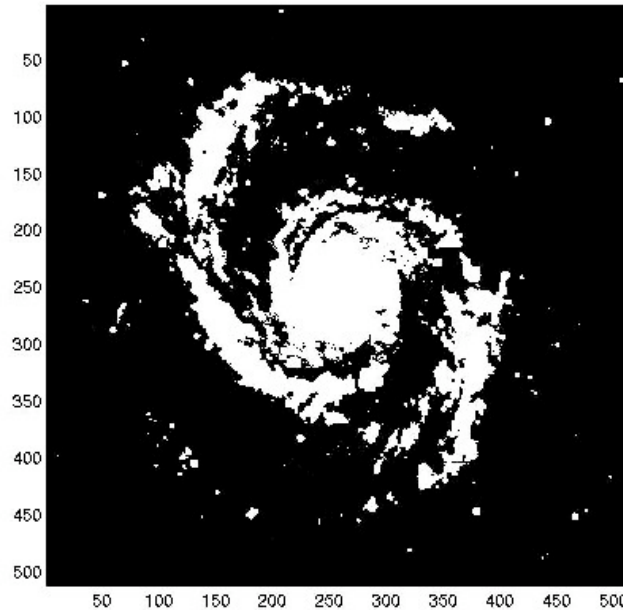
FD Analysis Results

Type	Domestic	Wild
Birch	1.262	1.113
Cherry	1.647	1.459
Oak	1.648	1.279
Maple	1.378	1.358
Dogwood	1.656	1.456
Poplar	1.424	1.456

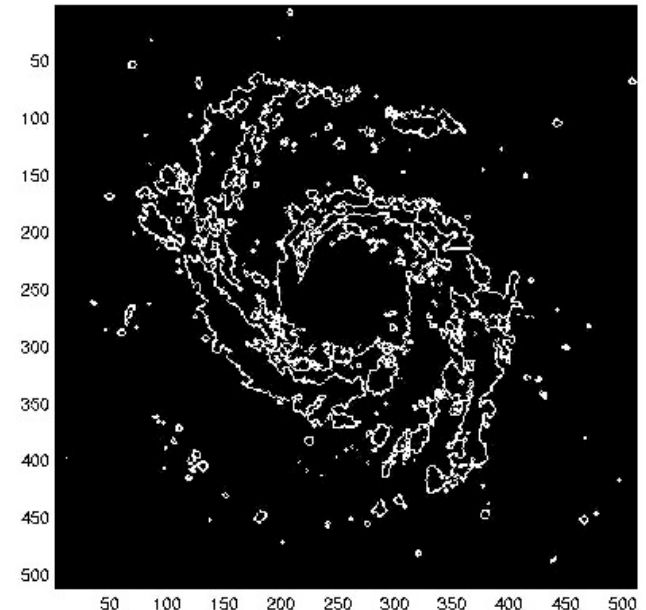
FD computation by box counting



Gray scale image

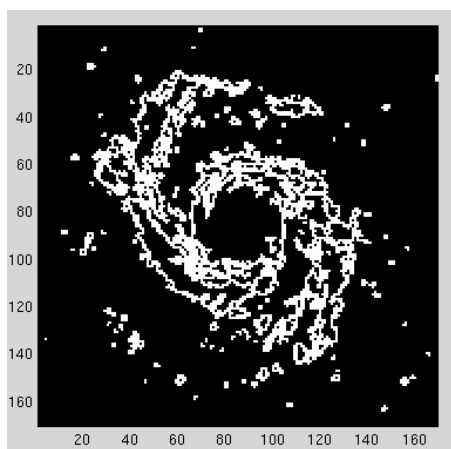


Binary image: pixels above a certain threshold are set to one

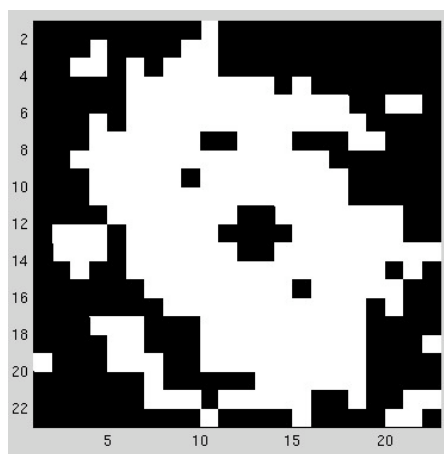


Boundary of the object is obtained

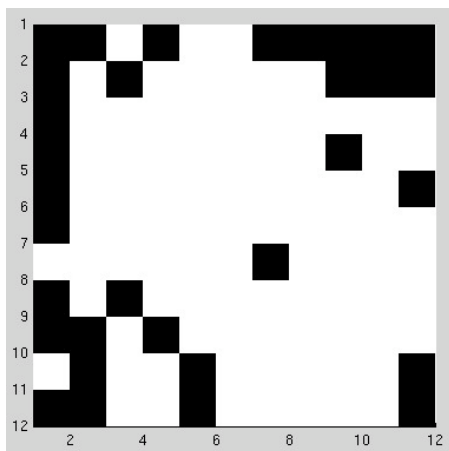
Box counting process



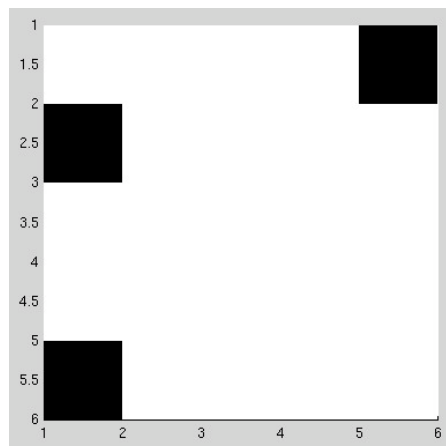
Box size 3 x 3



Box size 22 x 22



Box size 40 x 40



Box size 75 x 75

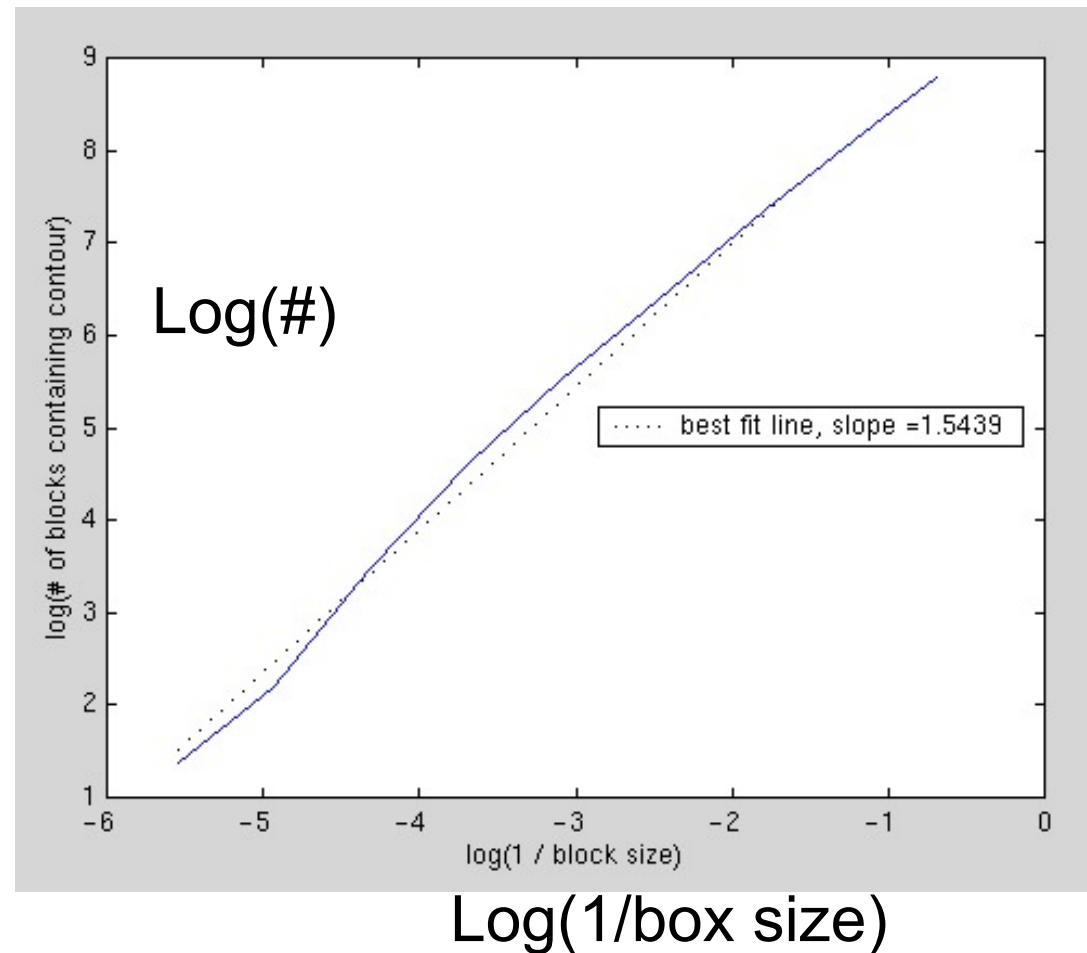
Break the image into boxes of a given size and count how many of those boxes contain the contour.

We perform this process for several different box sizes.

If a box contains the contour, it is colored white.

FD estimation

Box size (in pixels)	number of boxes containing the contour
2	6544
3	3897
6	1562
12	591
22	250
40	101
75	32
138	9
256	4



FD=slope =1.5439

Schedule for remaining classes

April 26 (Thursday). no class. Emotion symposium.

May 1 (Tuesday). Fractal analysis

May 3 (Thursday). Overview

May 8 (Tuesday). Student presentation.

May 10 (Thursday). Student presentation.

May 16 12:15pm. The deadline for research reports/literature review reports. After the deadline, reports will not be accepted (After the grade submission, I have to leave to LA immediately).

Schedule for Student presentation

Requirement for 3 credit students

15min presentation + 5 min discussions

Presentation will account for 20% of your final grade

- **Jamie L. Hanson** (VBM and effect of template)
- **Elizabeth B. Hutchinson** (?????)
- **Daniel L.B. Levinson** (VBM in what population ?)
- **David M. Perlman** (fMRI analysis on cortex ?)
- **Brianna S. Schuyler** (cortical thickness analysis)
- **Ammet B. Soni** (fMRI and machine learning)
- **Shubing Wang** (Fourier transform)
- **Daniel J. Kelley** (Brain asymmetry analysis)

Schedule for presentation

Please send PDF or PowerPoint to me via email in the morning of presentation to save time for switching laptops. If you go to PowerPoint print option, you can generate PDF (Apple laptop will be used and there are known conflict between Apple PowerPoint and Windows PowerPoint)

- **May 8**

11:00-11:20

11:20-11:40

11:40-12:00

12:00-12:20

- **May 10**

11:00-11:20

11:20-11:40

11:40-12:00

12:00-12:20