

Stat 311-Lecture 01, Probability

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1. Definition.

An *experiment* is any action whose outcome is subject to uncertainty (randomness).

The *sample space* S of an experiment is the set of all possible outcomes of the experiment.

An *event* A is a subset of the sample space, i.e. $A \subset S$.

Two events A and B are *mutually exclusive* (disjoint) if they have no outcomes in common, i.e. $A \cap B = \phi$.

2. Given a sample space S , probability P is a function that satisfies three axioms:

(i) $P(A) \geq 0$,

(ii) $P(S) = 1$,

(iii) For disjoint events A_1 and A_2 , $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

3. Problem.

You flip a biased coin repeatedly until you obtain a head. The probability of obtaining a head is $p > 0$ at each toss. Then we can show that $P(S) = 1$.

4. Using the axioms of probability, we can show the following theorems

(i) If $A \subset B$, $P(A) \leq P(B)$.

(ii) $P(A) = 1 - P(A^c)$.

(iii) If A and B are mutually exclusive, then

$$P(A \cap B) = 0.$$

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

5. Problem.

Prove

$$P(A \cap B) \geq P(A) + P(B) - 1,$$

when A, B are two events in the same sample space.

proof. We have shown in the class that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

But $P(A \cup B) \leq 1$. So by rearranging the terms, $P(A \cap B) \geq P(A) + P(B) - 1$. This is called the *Bonferroni inequality*.

6. Problem.

Show $P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)$.

proof. $P(A_1 \cup B) = P(A_1) + P(B) - P(A_1 \cap B) \leq P(A_1) + P(B)$. Now let $B = A_2 \cup A_3$. Then similarly $P(B) \leq P(A_2) + P(A_3)$. So $P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(B) \leq P(A_1) + P(A_2) + P(A_3)$.

7. Problem.

For events A and B , find formulas for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$ and $P(A \cap B)$ only.

(i) either A or B but not both

(ii) at most one of A or B

solution. (i) $P((A \cup B) \setminus (A \cap B)) = P(A \setminus B) + P(B \setminus A) = P(A) + P(B) - 2P(A \cap B)$.

(ii) Let D be the event of *at most one of A or B* . The complement of D is the event of *both A and B* , i.e. $D^c = A \cap B$. So $P(D) = 1 - P(D^c) = 1 - P(A \cap B)$.

8. Two players A and B flip a biased coin alternately and the first player to obtain a head wins. The probability of obtaining a head is $p > 0$ at each toss. If A flips first, find the probability that A wins the game.

Solution. The event of A wins the game can be decomposed into

$$A \text{ win} = \bigcup_{i=1}^{\infty} \{A \text{ wins on the } i\text{-th toss}\}.$$

Then we have

$$\begin{aligned} P(A \text{ wins}) &= p + (1-p)^2 p + (1-p)^4 p + \dots \\ &= \frac{p}{1 - (1-p)^2}. \end{aligned}$$