

Stat 311-Lecture 02, Combinatorics

Sept 13, 2005

- Given n distinguishable objects $x_1, \dots, x_n$, *permutation* is a rearrangement of the n objects. There are $n!$ different rearrangements.

Combination is a way to choose k objects out of the n distinct objects. There are $\binom{n}{k}$ different combinations. The notation is called n choose k .

- Binomial identities*

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

$$\sum_{k=0}^n n \binom{n}{k} = n2^{n-1}.$$

- Problem.* A fair coin is tossed independently n times ($n > 3$). Find the probability that we get 2 heads and $n - 2$ tails.
- Problem.* A fair coin is tossed independently n times ($n > 3$). Find the probability at least three of tosses yield heads.

Solution.

$$\begin{aligned} P(\text{at least 3 H's}) &= 1 - P(\text{at most 2 H's}) \\ &= 1 - \frac{1}{2^n} - \binom{n}{1} \frac{1}{2^n} - \binom{n}{2} \frac{1}{2^n}. \end{aligned}$$

- If n identical balls are placed at random into n distinct boxes, find the probability that exactly one box remains empty.

Solution. The size of the sample space is $\binom{n+n-1}{n}$ since it is equivalent to the number of ways of arranging n 0's and $n - 1$ 1's.

Let A be the empty box, B the box containing two balls and C's $n - 2$ boxes containing one ball. There are $n!/(n - 2)!$ ways of arranging the letter sequence $ABC \dots C$. So the probability is $\frac{n(n-1)}{\binom{2n-1}{n}}$.

- If n identical balls are placed into $n - 1$ distinct boxes so that each distinguishable arrangement is equally likely. Find the probability that no box remains empty.

Solution. The size of the sample space is $\binom{n+n-2}{n}$

since it is equivalent to the number of ways of arranging n 0's and $n - 2$ 1's.

Let A be the box containing two balls and B's $n - 2$ boxes containing one ball. There are $n - 1$ ways of arranging the letter sequence $AB \dots B$. So the probability is $\frac{n-1}{\binom{2n-2}{n}}$.

Homework 1. Due date Sept 20 8:00am. Since most of problems has the answers in the textbook, the homework will be graded based on the correctness of your derivation rather than the correctness of your answer. Solve the following 8 problems. Chapter 1, Problem 16., 17., 20. Theoretical Exercise 5; Chapter 2, Problem 2., 8., 45., Theoretical Exercise 19.