

Stat 311-Lecture 04, Independence

Sept 15, 2005

1. *Additional problem on conditional probability.* In a bolt factory, machines 1, 2 and 3 respectively produce 20 %, 30% and 50% of the total output. Of their output, 5%, 3% and 2% are defective. A bolt is selected random. (a) What is the probability that it is defective? (b) Given that it is defective, what is the probability that it was made by machine 1?
Solution. Let D be the event that the bolt is defective and M_1, M_2, M_3 be the events that the selected bolt comes from machines 1, 2 and 3 respectively. $P(M_1) = 0.2, P(M_2) = 0.3, P(M_3) = 0.5$. From the law of total probability,

$$P(D) = \sum_{i=1}^3 P(D|M_i)P(M_i),$$

where $P(D|M_1) = 0.05, P(D|M_2) = 0.03, P(D|M_3) = 0.02$. Hence, $P(D) = 0.029$.
 $P(M_1|D) = \frac{P(D \cap M_1)}{P(D)} = \frac{P(D|M_1)P(M_1)}{P(D)} = 0.52$.

2. Two events A and B are *independent* if $P(A|B) = P(A)$.

Theorem. A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

3. *Theorem.* If A and B are independent events, A^c and B are also independent.

Solution 1. Since A and B are independent, $P(A \cap B) = P(A)P(B)$. We need show $P(A^c \cap B) = P(A^c)P(B)$. $P(B) = P(A \cap B) + P(A^c \cap B) = P(A)P(B) + P(A^c \cap B)$. Hence $P(A^c \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B)$.

Solution 2. $P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = \frac{P(B) - P(A)P(B)}{P(B)} = 1 - P(A) = P(A^c)$.

4. Suppose that $A \subset B$ and $P(A) > 0$ and $P(B) < 1$. Are two events A and B independent?

Solution. Since $A \subset B$, $P(A \cap B) = P(A)$. The condition for the independence is $P(A \cap B) = P(A)P(B)$. $P(B) = P(A)P(B)$ iff. $P(B) = 1$. Hence, if $P(B) = 1$, A and B are independent but if $P(B) < 1$, A and B are not independent.

5. Suppose that A and B are independent events such that the probability that neither occurs is $\frac{1}{2}$ and the probability of B occurring is $\frac{1}{3}$. Determine the probability of A occurring.

Solution. $P((A \cup B)^c) = 1 - P(A \cup B) = 1/2$. So $P(A \cup B) = 1/2$. Since A and B are independent $P(A \cap B) = P(A)P(B) = P(A)/3$. By substituting terms into $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we have $1/2 = P(A) + 1/3 - P(A)/3$. Hence $P(A) = 1/4$.

6. If A and B are independent events, with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, find the following:

(a) $P(A^c \cap B^c)$

Solution 1. Since A and B are independent, A^c and B^c are independent. So $P(A^c \cap B^c) = P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = (1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2}$.

Solution 2. $P(A \cap B) = P(A)P(B) = \frac{1}{12}$. $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (\frac{1}{3} + \frac{1}{4} - \frac{1}{12}) = \frac{1}{2}$.

(b) $P(A^c|B)$.

Solution. Since A and B are independent, A^c and B are also independent. So $P(A^c|B) = P(A^c) = 1 - P(A) = \frac{2}{3}$.