

Stat 311-Lecture 06, Expectation

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1. The *expected value* of a discrete random variable X is defined as $\mathbb{E}X = \sum_{x \in S} xp(x)$.

2. *Problem.* A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the expected number of trials until he opens the door if unsuccessful keys are eliminated.

solution. Let X be the number of trials. Then $P(X = 1) = \frac{1}{n}, P(X = 2) = \frac{n-1}{n} \frac{1}{n-1} = \frac{1}{n}, \dots, P(X = n) = \frac{n-1}{n} \dots \frac{1}{2} = \frac{1}{n}$. The random variable X is said to be *uniformly distributed* (or a discrete uniform random variable).

$$\mathbb{E}X = \sum_{x=1}^n \frac{x}{n} = \frac{1}{n} \sum_{x=1}^n x = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

3. *Geometric distribution.* Toss a coin until the first head is obtained. Let X be the number of tosses. The probability function of X is given by $P(x) = (1-p)^{x-1}p$. This is called the geometric distribution. $\mathbb{E}X = 1/p$.

4. Two players A and B flip a biased coin alternately and the first player to obtain a head wins. The probability of obtaining a head is $p > 0$ at each toss. If A flips first, find the expected number of tosses that A must make before A wins.

solution. Let X be the number of tosses A makes until the game terminates. A wins the game if $H, TTH, TTTTH, \dots$ occur. Then the probability distribution for X is $p(x) = (1-p)^{2x-2}p, x = 1, 2, \dots$. So the expectation is $\mathbb{E}X = \sum_{x=1}^{\infty} x(1-p)^{2x-2}p$. Let $q = 1-p$. Then

$$\begin{aligned} \mathbb{E}X &= \frac{p}{2q} \sum_{x=1}^{\infty} 2xq^{2x-1} = \frac{p}{2q} \frac{d}{dq} \sum_{x=1}^{\infty} q^{2x} \\ &= \frac{p}{2q} \frac{d}{dq} \left(\frac{q^2}{1-q^2} \right) = \frac{1}{p(2-p)^2}. \end{aligned}$$

5. The expected value of function $Y = h(X)$ is

$$\begin{aligned} \mathbb{E}h(X) &= \sum_y yP(h(X) = y) \\ &= \sum_x h(x)P(h(X) = h(x)) \\ &= \sum_x h(x)P(X = x) = \sum_x h(x)p(x). \end{aligned}$$

6. Theorem. $\mathbb{E}(aX + b) = a\mathbb{E}X + b$.

7. For $X \sim \text{Binomial}(n, p)$, $\mathbb{E}X = np$.

$$\mathbb{E}X(X-1) = n(n-1)p^2.$$

8. *Problem.* Find a value c that minimizes $\mathbb{E}(X-c)^2$ for a discrete random variable X .

solution. Let $f(c) = \mathbb{E}(X-c)^2 = c^2 - (2\mathbb{E}X)c + \mathbb{E}X^2$. Then $f(c)$ is minimum at a point where $f'(c) = 0$. $f'(c) = 2c - 2\mathbb{E}X = 0$. So $c = \mathbb{E}X$ will give the minimum.

9. The variance of random variable X is

$$\mathbb{V}X = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

$$\mathbb{V}(aX + b) = a^2\mathbb{V}X.$$

10. A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean and the variance of the number of trials if unsuccessful keys are *not* eliminated.

solution. Let X be the number of trials until he opens the door. Note that $P(X = 1) = \frac{1}{n}, P(X = 2) = \frac{n-1}{n} \frac{1}{n}, P(X = 3) = \left(\frac{n-1}{n}\right)^2 \frac{1}{n}$. In general, $P(X = i) = \left(\frac{n-1}{n}\right)^{i-1} \frac{1}{n}$. So X is a geometric random variable with $p = \frac{1}{n}$. The expectation $\mathbb{E}X = \frac{1}{p} = n$. $\mathbf{Var}X = \frac{1-p}{p^2} = n^2 - n$.