

Stat 311-Lecture 08 Discrete Random Variables

Oct 4, 2005

1. *Negative binomial distribution with parameter r and p .* Let X be the number of coin tossing until r number of heads are accumulated. Then

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, \dots$$

The geometric distribution is a special case of a negative binomial distribution with $r = 1$.

$\mathbb{E}X^k = \frac{r}{p} \mathbb{E}(Y-1)^{k-1}$ where Y is a negative binomial with parameter $r+1$ and p . By letting $k = 1$, $\mathbb{E}X = \frac{r}{p}$. By letting $k = 2$, $\mathbb{E}X^k = \frac{r}{p} \left(\frac{r+1}{p} - 1 \right)$. So $\mathbb{V}X = \frac{r(1-p)}{p^2}$.

2. *The Banach match problem.* A mathematician carries two match boxes each containing n matches. The probability of using each match box is $1/2$. If he found one of the box is empty, what is the probability that the one box contains k matches?

Solution. Let A_k be the above event. Compute $P(A_n), P(A_{n-1}), \dots$ and you will see the pattern. $P(A_k) = 2 \binom{2n-k}{n} \left(\frac{1}{2}\right)^{2n-k}, k = 0, 1, \dots, n$.

3. *Hypergeometric distribution.* A box contains N balls, of which m are white and $N - m$ are black. Let X be the number of white balls in n draws without replacement.

$$p(x) = \frac{\binom{m}{x} \binom{N-m}{n-x}}{\binom{N}{n}}, x = 0, 1, \dots, \min(n, m).$$

4. Two people toss a fair coin n times each. Find the probability that they throw equal number of heads.
solution. Let E_i be the event the both throws i heads. Then we are computing

$$P(\cup_{i=0}^n E_i) = \sum_{i=0}^n P(E_i).$$

Note $P(E_i) = \binom{n}{i} \left(\frac{1}{2}\right)^n \binom{n}{i} \left(\frac{1}{2}\right)^n$.

This problem can be also solved using random variables. Let X be the number of heads for the first person and Y be the number of heads for the

second person. Then $X, Y \sim i.i.d.$ Binomial($n, \frac{1}{2}$).

$$\begin{aligned} P(X=Y) &= \sum_{i=0}^n P(X=i, Y=i) \\ &= \sum_{i=0}^n P(X=i)P(Y=i) \\ &= \sum_{i=0}^n \binom{n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-i} \binom{n}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{n-i} \\ &= \frac{1}{2^{2n}} \sum_{i=0}^n \binom{n}{i} \binom{n}{i} \end{aligned}$$

From the hypergeometric distribution,

$$\binom{2n}{n} = \sum_{i=0}^n \binom{n}{i} \binom{n}{i}.$$

Hence, the probability is $\frac{\binom{2n}{n}}{2^{2n}}$.

NOTE: The first midterm exam will cover lectures 1-9 (upto Chapter 4). **HW 3** due Oct 20. Solve the following 9 problems. Chapter 4 Problems 42, 44, 48, 60, 64, 75, 76. Theoretical Exercises 31, 32.