

Stat 311-Lecture 09 Continuous Random Variables

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1. The cumulative distribution of X is defined as $F(x) = P(X \leq x)$. $F(-\infty) = 0, F(\infty) = 1$.

2. Problem. For $X \sim \text{Binomial}(n, p)$,

$$F(x) = \sum_{y=0}^i \binom{n}{y} p^y (1-p)^{n-y}$$

for integer $i \leq x \leq i+1$.

3. Problem. For positive integer valued discrete random variable X , show

$$\mathbb{E}X = \sum_{x=1}^{\infty} P(X \geq x).$$

Note $\sum_{x=1}^{\infty} P(X \geq x) = \sum_{x=1}^{\infty} \sum_{y=x}^{\infty} P(y) = \sum_{y=1}^{\infty} \sum_{x=1}^y P(y) = \sum_{y=1}^{\infty} y P(y) = \mathbb{E}X$.

4. The constant π is usually computed by summing up infinite series, i.e.

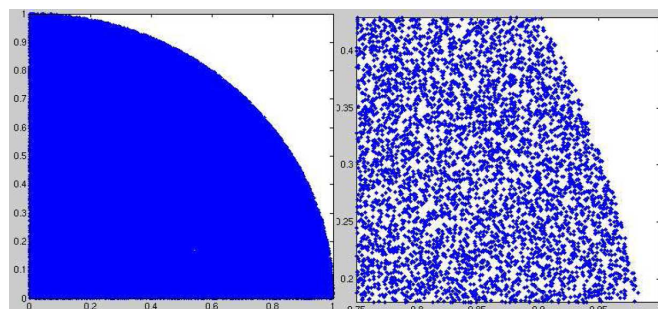
$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right) = 3.141592 \dots$$

The convergence for computing this series is very slow. There is a radically different way to compute the digits of π based on the theory of probability. Let A be the area of a unit circle bounded by a unit square B . Suppose we throw darts at the square at random. Let (X, Y) be the coordinates of the dart. It is a pair of random variables. X is continuously defined at every point of the continuous domain A . This type of random variable is called a *continuous* random variable. Now define binary random variable $Z = 1$ if a dart hits A and $Z = 0$ if it hits B/A . Now we define Z_i similarly for the i -th dart thrown. Then we can show

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n Z_i}{n} \rightarrow \pi.$$

The computer code MATLAB to simulate 1,000,000 dart throwing experiment.

```
x=rand(1000000,1);
y=rand(1000000,1);
rsquare=x.^2+y.^2;
```



```
accept=find(rsquare <1);
>> accept'
ans =
     2     3     4     5     6     7     8    10    11    15 ...
plot(x(accept),y(accept),'.')
>>4*size(accept,1)/1000000
ans =
     3.1336
```

5. Formally, X is a *continuous* random variable if there is nonnegative function f , called probability density, such that

$$P(X \in B) = \int_B f(x) dx.$$

for every $B \subset \mathbb{R}$. Let $B = [a, b]$. Then

$$P(a \leq X \leq b) = \int_a^b f(x) dx.$$

By letting $a = b$,

$$P(a \leq X \leq a) = \int_a^a f(x) dx = 0.$$

By letting $a = -\infty$ and $b = \infty$,

$$1 = P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f(x) dx = 0.$$

6. Let X have the probability density function $f(x) = c|x|$ for $-1 < x < 1$ and $f(x) = 0$ otherwise, where $|x|$ denotes the absolute value of x . Find the value c . *solution.* To find c , we integrate

$$\int_{-1}^1 c|x| dx = \int_{-1}^0 -cx dx + \int_0^1 cx dx = 1.$$

So $c = 1$.