

# Stat 311-Lecture 11 Central Limit Theorem

Oct 24, 2005

1. If the cdfs of  $X$  and  $Y$  are identical, two random variables are identically distributed. This does not imply  $X = Y$  which is nonsense. To denote the equality of distribution, we will use notation  $X \sim Y$ .
2. If  $S_n = X_1 + X_2 + \cdots + X_n$ , where  $X_i$  are identically distributed random variables coming from independent events with  $\mathbb{E}X_i = \mu$  and  $\mathbb{V}X_i = \sigma^2$ .  $X_1, X_2, \dots, X_n$  are usually called i.i.d. random variables. Let

$$Z_n = \frac{S_n - \mathbb{E}S_n}{\sqrt{\mathbb{V}S_n}} = \frac{S_n - n\mu}{\sqrt{n}\sigma}.$$

Then for large  $n$ ,

$$P(Z_n \leq x) \approx \Phi(x),$$

where  $\Phi$  is the cdf for a standard normal distribution. Note

$$P(a \leq Z_n \leq b) \approx \Phi(b) - \Phi(a).$$

3. Problem. Let  $X$  be the number of heads in 40 tossed coins. Find the probability  $X = 20$ . Solution. Note  $X = X_1 + X_2 + \cdots + X_{40}$  with  $X_i \sim \text{Bernoulli}(0.5)$ . Note  $\mathbb{E}X = 40 \cdot 0.5$ ,  $\mathbb{V}X = 40 \cdot 0.5^2$ . Let  $S = \frac{X-20}{\sqrt{10}}$ .

$$\begin{aligned} P(X = 20) &= P(19.5 \leq X \leq 20.5) \\ &= \Phi\left(\frac{0.5}{\sqrt{10}}\right) - \Phi\left(-\frac{0.5}{\sqrt{10}}\right) \\ &= 2\Phi\left(\frac{0.5}{\sqrt{10}}\right) - 1 = 20.5636 - 1 = 0.1272. \end{aligned}$$

The exact result is  $P(X = 20) = \binom{40}{20} 0.5^{40} = 0.1254$ .

4. Problem. A fair coin is thrown 1000 times. Find the approximate probability that the total number of heads among 1000 tosses will lie between 400 and 600 using the Central Limit Theorem.

Solution. Let  $X_i \sim \text{Bernoulli}(\frac{1}{2})$ . With notation  $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$ ,  $P(400 \leq \sum_{i=1}^{1000} X_i \leq 600) = P(-0.1 \leq \bar{X} - 0.5 \leq 0.1) = P\left(\frac{-0.1}{1/20\sqrt{10}} \leq \frac{\bar{X}-0.5}{1/20\sqrt{10}} \leq \frac{0.1}{1/20\sqrt{10}}\right) = 2\Phi\left(\frac{2}{\sqrt{10}}\right) - 1 = 0.47$ .

5. Problem. The expected service time for a customer coming through a checkout counter in a retail store is 2 minutes while its variance is 1. (a) Approximate the probability that 100 customers can be served in less than 3 hours of total service time. (b) Find the number of customers that can be served in less than 3 hours with probability 0.9.

Solution. (a) Let  $X_i$  be the service time for the  $i$ -th customers.  $S = X_1 + \cdots + X_{100}$ .  $\mathbb{E}S = 200$ ,  $\mathbb{V}S = 100$ . Let  $Z = \frac{S-200}{10}$ . Then

$$P(S \leq 180) = \Phi\left(\frac{180-200}{10}\right) = 1 - \Phi(2) = 0.0228$$

(b) For  $S_n = X_1 + \cdots + X_n$ ,  $\mathbb{E}S = 2n$ ,  $\mathbb{V}S = n$ . Let  $Z = \frac{S_n-2n}{\sqrt{n}}$ . Then we need  $P(Z \leq \frac{180-2n}{\sqrt{n}}) = 0.9$ . From the table  $\Phi(1.28) = 0.9$ . So we need to solve  $180 - 2n = 1.28\sqrt{n}$ .

**HW 3** Due Nov. 10. 8:00AM. Solve the following 8 problems in Chapter 5. Problems 1, 10, 11, 16, 26, 37 Theoretical Exercises 26, 29.

**Exam schedule:** Nov. 17 (Thursday). Second midterm exam will cover chapters 4-5 and parts of chapter 6 up to November 10th lecture. We will have 4-5 problems.