

Stat 311-Lecture 12 Other Continuous Random Variables

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1. Exponential random variable. $X \sim \exp(\lambda)$. Density function: $f(x) = \lambda e^{-\lambda x}, x \geq 0$. c.d.f. $F(x) = 1 - \exp(-\lambda x), x \geq 0$. $\mathbb{E}X^n = \frac{n}{\lambda} \mathbb{E}X^{n-1}$. $\mathbb{E}X = \frac{1}{\lambda}, \mathbb{V}X = \frac{1}{\lambda^2}$.

2. Problem. The density function of an exponential random variable X is $f(x) = \frac{1}{\beta} e^{-x/\beta}$ for $x > 0$. Assume that the mean value of X is 1. (a) Find the cumulative distribution function of X and plot its graph. (b) Derive the probability density function of $Y = 2X$ using the cumulative distribution function of X .

3. Problem. Let X have the probability density function $f(x) = ce^{-x}$ for $1 < x < \infty$ and $f(x) = 0$ otherwise.

(a) Find the value c .

Solution. $\int_1^\infty ce^{-x} dx = -ce^{-x}|_1^\infty = ce^{-1} = 1$. So $c = e$.

(b) Find the cumulative distribution function of X and plot its graph.

Solution. Integrating f , we get $F(x) = 0$ for $x \leq 1$ and $F(x) = 1 - e^{1-x}$ for $x > 1$. F is a nondecreasing function and $F(\infty) = 1$.

(c) Find $P(-1 < X \leq 3 | X \geq 3)$.

Solution. $P(-1 < X \leq 3 | X \geq 3) = \frac{P(X=3)}{P(X \geq 3)} = 0$ since the probability that a continuous random variable taking a single point is zero.

4. Gamma random variable. $X \sim \text{Gamma}(\alpha, \lambda)$.

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, x \geq 0.$$

Note the gamma function

$$\Gamma(\alpha) = \int_0^\infty \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} dx = (\alpha-1)\Gamma(\alpha-1).$$

For integer values, $\Gamma(\alpha) = (\alpha-1)!$.

$$\mathbb{E}X = \frac{\alpha}{\lambda}, \mathbb{V}X = \frac{\alpha}{\lambda^2}.$$

Special cases. $\text{Gamma}(1, \lambda) \sim \exp(\lambda)$ and $\text{Gamma}(\frac{n}{2}, \frac{1}{2}) \sim \chi_n^2$, chi-squared distribution with n degrees of freedom. If $Z_i \sim N(0, 1)$, $Z_1^2 + \dots, Z_n^2 \sim \chi_n^2$.

5. Problem. The lifetime (in hours) of an electronic component is a random variable with density function given by

$$f(x) = e^{-x}, x > 0.$$

Four of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment will operate for at least 2 hours without failure.

Solution. The probability of single component lasting more than 2 hours is $p = \int_2^\infty e^{-x} dx = e^{-2}$. Then we are seeking

$$\binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4.$$

6. If X and Y are continuous random variables with joint c.d.f. $F(x, y)$, express $P(a \leq X \leq b, c \leq Y \leq d)$ using F .

Solution. Draw $\{a \leq X \leq b, c \leq Y \leq d\} \subset \mathbb{R}^2$. Then $P(a \leq X \leq b, c \leq Y \leq d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$ using the fact that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

7. Suppose that the joint probability density function f of X and Y is

$$f(x, y) = \begin{cases} k(x^2 + y) & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value k and the marginal density functions. Are X and Y independent random variables? Justify your answer.

Solution. $\int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) dx dy = \int_0^2 \int_0^1 k(x^2 + y) dx dy = \int_0^2 k(\frac{1}{3} + y) dy = \frac{8}{3}k = 1$. So $k = \frac{3}{8}$.

$$f_X(x) = \int_0^2 \frac{3}{8}(x^2 + y) dy = \begin{cases} \frac{3}{4}(x^2 + 1) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^1 \frac{3}{8}(x^2 + y) dx = \begin{cases} \frac{1}{8}(1 + 3y) & \text{for } 0 \leq y \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(0, 1) \neq f_X(0)f_Y(1)$, they are not independent.