

Stat 311-Lecture 14 Conditional distributions

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1. The *conditional probability mass function* of Y given $X = x$ is given by

$$p(y|x) = \frac{p(x, y)}{p_X(x)}.$$

The conditional expectation of Y given $X = x$ is then

$$\mathbb{E}(Y|X = x) = \sum_y yp(y|x).$$

If X and Y are independent, $\mathbb{E}(Y|X = x) = \mathbb{E}Y$.

2. Problem. Suppose $X, Y \sim \text{Binomial}(n, \frac{1}{2})$ and they are independent. Find $\mathbb{E}(X|X + Y = z)$.

Solution. We need to compute the conditional distribution first. Note

$$Z = X + Y \sim \text{Binomial}(2n, \frac{1}{2}).$$

Then the conditional probability mass function is given by

$$\begin{aligned} p(x|z) &= \frac{P(X = x, Y = z - x)}{P(X + Y = z)} \\ &= \frac{\binom{n}{x} \binom{n}{z-x}}{\binom{2n}{z}}. \end{aligned}$$

This is a hypergeometric distribution.

$$\begin{aligned} \mathbb{E}[X|X + Y = z] &= \sum_{x=0}^z xp(x|z) \\ &= \sum_{x=0}^z x \frac{\binom{n}{x} \binom{n}{z-x}}{\binom{2n}{z}} = \sum_{x=0}^z \frac{n \binom{n-1}{x-1} \binom{n}{z-x}}{z \binom{2n-1}{z-1}} = \frac{z}{2}. \end{aligned}$$

3. The *conditional probability density* of Y given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}.$$

The *conditional expectation* of Y given $X = x$ is

$$\mathbb{E}(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy.$$

4. Problem. Given $f(y_1, y_2) = \frac{1}{y_1}$ for $0 \leq y_2 \leq y_1 \leq 1$ and 0 otherwise, determine $P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2}) = \frac{1}{8}$. $P(Y_1 \geq \frac{1}{2} | Y_2 = 4) = \frac{1}{2}$.

5. Let X_1, X_2 be i.i.d. $\text{Uniform}(0, a)$ and let $Y_1 = \min(X_1, X_2)$, $Y_2 = \max(X_1, X_2)$. Find the joint probability density of Y_1 and Y_2 .

solution. First note that $0 < y_1 < y_2 < a$ is the domain for the joint density. The joint c.d.f. of Y_1 and Y_2 is

$$\begin{aligned} F_{Y_1, Y_2}(y_1, y_2) &= P(Y_1 \leq y_1, Y_2 \leq y_2) \\ &= P(Y_2 \leq y_2) - P(Y_1 > y_1, Y_2 \leq y_2) \\ &= P(X_1 \leq y_2)P(X_2 \leq y_2) \\ &\quad - P(y_1 < X_1 < y_2)P(y_1 < X_2 < y_2) \\ &= \left(\frac{y_2}{a}\right)^2 - \left(\frac{y_2 - y_1}{a}\right)^2, 0 < y_1 < y_2 < a. \end{aligned}$$

Then the joint density is $f_{Y_1, Y_2}(y_1, y_2) = \frac{\partial^2}{\partial y_1 \partial y_2} F_{Y_1, Y_2}(y_1, y_2) = \frac{2}{a^2}, 0 < y_1 < y_2 < a$. So it is a uniform distribution over a triangle $0 < y_1 < y_2 < a$.