

Stat 311-Lecture 14 Moment generating functions

Dec 5, 2005

- For random variable X , the moment generating function of X is a function of t given by

$$M_X(t) = \mathbb{E}e^{tX}$$

for any $|t| \leq c$. This function can be used in determining any n -th moment automatically. The Taylor expansion of e^{tX} is

$$e^{tX} = 1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 \dots$$

Taking the expectation on the both side, we have

$$M_X(t) = 1 + t\mathbb{E}X + \frac{t^2}{2!}\mathbb{E}X^2 + \frac{t^3}{3!}\mathbb{E}X^3 \dots$$

Hence the series expansion of the moment generating function consists of the moments of X . By differentiating the moment generating function and letting $t = 0$, we have

$$M'_X(0) = \mathbb{E}X,$$

$$M''_X(0) = \mathbb{E}X^2.$$

So the variance can be obtained as

$$\mathbb{V}X = M''_X(0) - (M'_X(0))^2.$$

- For $X_i \sim \text{Bernoulli}(p)$, the moment generating function is given by $M_{X_i}(t) = 1 - p + pe^t$.
- For iid $X_1, X_2, \dots, X_n$, the moment generating functions of $S_n = X_1 + X_2 + \dots + X_n$ is given by

$$M_{S_n}(t) = \mathbb{E}e^{t(X_1+X_2+\dots+X_n)} = M_{X_1}^n(t).$$

Using this property, the moment generating for $X \sim \text{Binomial}(n, p)$ is given by $M_X(t) = (1 - p + pe^t)^n$. Note that $M'_X(t) = n(1 - p + pe^t)^{n-1}pe^t$. So $\mathbb{E}X = M'_X(0) = np$.

- Random variables X and Y have the same distribution if $M_X(t) = M_Y(t)$ for all possible values of t . This property can be used to show that the sum of independent binomial distributions is again binomial. Consider independent $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$. The moment generating functions are $M_X(t) = ((1 - p + pe^t)^{n_1})$ and $M_Y(t) = ((1 - p + pe^t)^{n_2})$. The moment generating function for $X + Y$ is

$$M_{X+Y}(t) = M_X(t)M_Y(t) = (1-p+pe^t)^{n_1}(pe^t+q)^{n_2}.$$

Since this is the moment generating function of $\text{Binomial}(n_1 + n_2, p)$, $X + Y$ must be distributed as $\text{Binomial}(n_1 + n_2, p)$.

- Suppose that X and Y are independent and identically distributed random variables whose density is of the form $f_X(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$, $\alpha > 0, x > 0$. Evaluate the n -th moment of X and find the density of $X + Y$.
solution. X and Y are gamma distributions. Directly we can compute

$$\begin{aligned} \mathbb{E}X^n &= \int_0^\infty x^n \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} dx \\ &= \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \int_0^\infty \frac{x^{n+\alpha-1}e^{-x}}{\Gamma(n+\alpha)} dx = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}. \end{aligned}$$

Using the moment generating function method, we have

$$M_X(t) = \int_0^\infty e^{tx} \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} dx = \int_0^\infty \frac{x^{\alpha-1}e^{-(1-t)x}}{\Gamma(\alpha)} dx.$$

Let $y = (1 - t)x$. Then

$$\begin{aligned} M_X(t) &= \int_0^\infty \left(\frac{y}{1-t}\right)^{\alpha-1} \frac{e^{-y}}{\Gamma(\alpha)} \frac{dy}{1-t} \\ &= \frac{1}{(1-t)^\alpha} \int_0^\infty \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy = \frac{1}{(1-t)^\alpha}. \end{aligned}$$

Since X and Y are i.i.d., $M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{1}{(1-t)^{2\alpha}}$, indicating that the density of $X + Y$ should have the same form as the density of X with different parameter 2α . Hence,

$$f_{X+Y}(v) = \frac{v^{2\alpha-1}e^{-v}}{\Gamma(2\alpha)}, 0 < v.$$

The sum of independent gamma distributions are gamma.

- For independent $X, Y \sim N(\mu, \sigma^2)$, proving the normality of $\alpha X + \beta Y$ is similar to the previous example.