

STAT 311 2005 fall semester midterm problems and answers

1. 10% of the glass bottles coming off a production line have serious flaws in the glass. If two bottles are randomly selected, find the mean and variance of the number of bottles that have serious flaws by directly computing them using the probability mass function. The answer should be correct up to 3 decimal places.
Solution. Let X be the number of flawed bottles. The probability mass function of X is $p(0) = 0.1^2, p(1) = 2 \cdot 0.9 \cdot 0.1, p(2) = 0.1^3$. $\mathbb{E}X = p(1) + 2p(2) = 2 \cdot 0.9 \cdot 0.1 + 2 \cdot 0.1^3 = 0.182$. $\mathbb{E}X^2 = p(1) + 2^2p(2) = 2 \cdot 0.9 \cdot 0.1 + 2^2 \cdot 0.1^3 = 0.184$. $\mathbb{V}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 0.151$.

2. A lie detector will show a positive reading (indicates a lie) 10% of the time when a person is telling the truth and 90% of the time when the person is lying. Suppose two people are suspects in a crime and only one is guilty. The guilty one will be lying while the innocent one will be telling the truth.

(a) What is the probability that the detector shows positive readings for both suspects?

Solution.

$$\begin{aligned} P(\text{positive}) &= P(\text{positive} \cap \text{lying}) + P(\text{positive} \cap \text{truth}) \\ &= P(\text{positive}|\text{lying})P(\text{lying}) + P(\text{positive}|\text{truth})P(\text{truth}) \\ &= 0.1 \times 0.5 + 0.9 \times 0.5 = 0.5 \end{aligned}$$

(b) What is the probability that a person is lying when the detector shows a negative reading?

Solution.

$$\begin{aligned} P(\text{lying}|\text{negative}) &= \frac{P(\text{negative}|\text{lying})P(\text{lying})}{P(\text{negative})} \\ &= \frac{P(\text{negative}|\text{lying})P(\text{lying})}{P(\text{negative}|\text{lying})P(\text{lying}) + P(\text{negative}|\text{truth})P(\text{truth})}. \end{aligned}$$

$$P(\text{negative}|\text{lying}) = 1 - P(\text{positive}|\text{lying}) = 1 - 0.9 = 0.1.$$

$$P(\text{negative}|\text{truth}) = 1 - P(\text{positive}|\text{truth}) = 1 - 0.1 = 0.9. \text{ Hence}$$

$$P(\text{lying}|\text{negative}) = \frac{0.1 \cdot 0.5}{0.1 \cdot 0.5 + 0.9 \cdot 0.5} = 0.1.$$

3. (a) Prove $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$.
 (b) Prove $P(A \cap B \cap C) \geq P(A) + P(B) + P(C) - 2$.
Solution. (a) $P(A \cup D) = P(A) + P(D) - P(A \cap D) \leq P(A) + P(D)$. Now let $D = B \cup C$. Then similarly $P(D) \leq P(B) + P(C)$. So $P(A \cup B \cup C) \leq P(A) + P(D) \leq P(A) + P(B) + P(C)$.
 (b) Note $P(A \cup D) = P(A) + P(D) - P(A \cap D) \leq 1$. So $P(A \cap D) \geq P(A) + P(D) - 1$. This inequality is called the *Bonferroni's inequality*. Now let $D = B \cap C$ and reapply the Bonferroni's inequality and we prove the statement.

4. (a) Consider equation $X_1 + X_2 + X_3 + X_4 = 0$. If each X_i can only take value either 1 or -1 , determine the number of unique solutions to this equation. Note $(X_1, X_2, X_3, X_4) = (1, -1, 1, -1)$ is a solution while $(X_1, X_2, X_3, X_4) = (1, -1, -1, -1)$ is not.

Assume further that X_i is a random variable such that $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$. Determine the probability that the equation has a solution.

(b) Consider equation $X_1 + X_2 + X_3 + \dots + X_8 = 0$. Each X_i can only take one of three values 1, 0 or -1 . The probability of X_i taking one of the three values is $\frac{1}{3}$. Determine the probability the equation has a solution.

Solution. (a) If 2 of variables are 1 and the remaining 2 are -1, we have solution. Hence there are $\binom{4}{2}$ possible solutions. The size of the sample space is 2^4 so the probability is $\frac{\binom{4}{2}}{2^4}$. (b) Let Y be the number of variables that take value 1 and Z be the number of variables that take value -1. Then $n - Z - Y$ is the number of variables that take value 0. In order to have a solution, we need $Y = Z$. $P(Y = Z) = \sum_{x=0}^4 P(Y = Z = x) = \sum_{x=0}^4 \binom{8}{x} \binom{8-x}{x} \left(\frac{1}{3}\right)^8$.