STAT 311 Midterm 2 solutions, 2005

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1. Two players A and B flip a fair coin alternately and the first player to obtain a head wins. Suppose A flips first.
   (a) Let X be the number of tosses player A is making till he wins the game. Determine the probability mass function of X. What is $\mathbb{E}X$?
   (b) What is the probability A wins the game with more than 4 tosses.

   Solution. (a) Let $p = \frac{1}{2}$. The probability mass function for X is given by $p(1) = p$, $p(2) = (1-p)^2p$, $p(3) = (1-p)^4p$, $p(4) = (1-p)^6p \cdots$, $p(x) = (1-p)^{2(x-1)}p$. $\mathbb{E}X = \sum_{x=1}^{\infty} x(1-p)^{2(x-1)}p$. Let $r = (1-p)^2$. Then

   $$\mathbb{E}X = \sum_{x=1}^{\infty} xr^{x-1} = \sum_{x=1}^{\infty} \frac{d}{dr} r^x = \frac{d}{dr} \sum_{x=1}^{\infty} r^x = \frac{d}{dr}\frac{r}{1-r} = \frac{1}{(1-r)^2} = \frac{1}{(1-1/4)^2}.$$

   (b) $P(X \geq 4) = 1 - P(X \leq 3) = 1 - p - (1-p)^2p - (1-p)^4p = 1 - 0.5 - 0.5^2 \cdot 0.5 - 0.5^4 \cdot 0.5 = 0.34$

2. 65% of the population of a large community is in favor of a proposed rise in school taxes.
   (a) Find the probability that 100 people randomly selected will contain fewer than 75 in favor.
   (b) Find the probability that 100 people randomly selected will contain more than 55 in favor.

   Solution. You will get the full credit even without the continuity correction. (a) Let $X_i = 1$ if the i-th person is in favor. Then $p = p(X_i = 1) = 0.65$. Let $X = X_1 + X_2 + \cdots X_{100}$ be the total number of people who favor. $X \sim Binomial(100, p)$. We apply the central limit theorem.

   $$P(X \leq 75) = P\Big(\frac{X - 100p}{\sqrt{100p(1-p)}} \leq \frac{75 - 100p}{\sqrt{100p(1-p)}}\Big) = \Phi(\frac{75 - 100p}{\sqrt{100p(1-p)}}) = \Phi(\frac{75 - 65}{\sqrt{65 \cdot 0.35}}) = \Phi(2.0966) = 0.9817$$

   (b) $P(X \geq 55) = P\Big(\frac{X-100p}{\sqrt{100p(1-p)}} \geq \frac{55-100p}{\sqrt{100p(1-p)}}\Big) = P(Z \geq -2.0966) = P(Z \leq 2.0966) = \Phi(2.0966) = 0.9817$.

3. The number of bacteria colonies in a sample of polluted water has a Poisson random variable with a mean of 2 per cubic centimeter. If 4 1-cubic-centimeter samples are independently collected from the same polluted water, find the probability that at least one sample will contain one or more bacteria colonies. Hint. The probability mass function of a Poisson random variable with parameter λ is $p(x) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0, 1, 2, .$

   Solution. Let X be the number of bacteria colonies in a sample. Note $p = P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda} = 1 - e^{-2}$ is the probability the i-th sample has one or more bacteria colonies. Now let Y be the number of samples that contain more than one or more bacteria colonies. Then $Y \sim Binomial(4, p)$. Then we need $P(Y \geq 1) = 1 - P(Y = 0) = 1 - \binom{4}{0}p^0(1-p)^4 = 1 - e^{-4\lambda} = 1 - e^{-8}$.

4. The probability density function of random variable X is given by $f(x) = e^{-x}, x \geq 0$.
   (a) Determine the probability density function of $Y = 2X + 1$.
   (b) Determine $\mathbb{E}Y$.
   Solution. (a) Note that $F_X(x) = \int_0^x e^{-x}\,dx = 1 - e^{-x}$ for $x \geq 0$. $F_Y(y) = P(2X + 1 \leq y) = F_X(\frac{y-1}{2}) = 1 - e^{-\frac{y-1}{2}}$ for $y \geq 1$. Then $f_Y(y) = \frac{1}{2}e^{-\frac{y-1}{2}}$ for $y \geq 1$.
   (b) $\mathbb{E}Y = 2\mathbb{E}X + 1 = 2\int_0^{\infty} xe^{-x}\,dx + 1$. Use the integration by parts so that $\int_0^{\infty} xe^{-x}\,dx = -xe^{-x} - e^{-x}|_0^{\infty} = 1$. So we have $\mathbb{E}Y = 3$.