

STAT 311 Old midterm problems and answers

1. If A and B are independent events, with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, find the following:

(a) $P(A^c \cap B^c)$

Solution 1. Since A and B are independent, A^c and B^c are independent. So $P(A^c \cap B^c) = P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = (1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2}$.

Solution 2. $P(A \cap B) = P(A)P(B) = \frac{1}{12}$. $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (\frac{1}{3} + \frac{1}{4} - \frac{1}{12}) = \frac{1}{2}$.

(b) $P(A^c|B)$.

Solution. Since A and B are independent, A^c and B are also independent. So $P(A^c|B) = P(A^c) = 1 - P(A) = \frac{2}{3}$.

2. In a bolt factory, machines 1, 2 and 3 respectively produce 20 %, 30% and 50% of the total output. Of their output, 5%, 3% and 2% are defective. A bolt is selected random. (a) List the sample space for this experiment and assign a probability to each sample point. (b) What is the probability that it is defective? (c) Given that it is defective, what is the probability that it was made by machine 1?

Solution. Let D be the event that the bolt is defective and M_1, M_2, M_3 be the events that the selected bolt comes from machines 1, 2 and 3 respectively. Then the sample space is given by $S = M_1 \cup M_2 \cup M_3$ with $P(M_1) = 0.2, P(M_2) = 0.3, P(M_3) = 0.5$. From the law of total probability,

$$P(D) = P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_3),$$

where $P(D|M_1) = 0.05, P(D|M_2) = 0.03, P(D|M_3) = 0.02$. Hence, $P(D) = 0.029$. $P(M_1|D) = \frac{P(D \cap M_1)}{P(D)} = \frac{P(D|M_1)P(M_1)}{P(D)} = 0.52$

3. A box contains 100 white and 200 black balls. Balls are randomly selected, one at a time, until a black one is obtained. Assume that each selected ball is replaced before the next one is drawn. (a) What is the probability that exactly n draws are needed? (b) What is the probability that at least 10 draws are needed? (c) What is the expected number of draws until a black ball is obtained?

Solution. Let X be the number of draws. Then $P(X = n) = (\frac{5}{15})^{n-1} \frac{10}{15} = (\frac{1}{3})^{n-1} \frac{2}{3}$. This is a geometric distribution with parameter $p = \frac{2}{3}$.

$$\begin{aligned} P(X \geq 10) &= \sum_{n=10}^{\infty} P(X = n) = \sum_{n=10}^{\infty} (\frac{1}{3})^{n-1} \frac{2}{3} = \frac{2}{3} (\frac{1}{3})^9 \sum_{n=0}^{\infty} (\frac{1}{3})^n \\ &= \frac{2}{3} (\frac{1}{3})^9 \frac{1}{1 - 1/3} = (\frac{1}{3})^9. \end{aligned}$$

4. The number of cars driving past a parking area in a 1-minute has Poisson distribution with parameter λ . The probability that any driver wish to park his car is p . Let D be the number of drives who wish to park during a 1-minute interval. Find its probability mass function.

Solution. Let X be the number of cars driving past a parking area. $X \sim \text{Poisson}(\lambda)$. Then

$$\begin{aligned} P(D = k) &= \sum_{i=k}^{\infty} P(D = k \text{ and } X = i) \\ &= \sum_{i=k}^{\infty} P(D = k|X = i)P(X = i) \\ &= \sum_{i=k}^{\infty} \left(\binom{i}{k} p^k (1-p)^{i-k} \right) \left(e^{-\lambda} \frac{\lambda^i}{i!} \right) \\ &= \frac{(\lambda p)^k e^{-\lambda}}{k!} \sum_{i=k}^{\infty} \frac{[(1-p)\lambda]^{i-k}}{(i-k)!} = \frac{(\lambda p)^k e^{-\lambda}}{k!} e^{(1-p)\lambda} = \frac{(\lambda p)^k e^{-\lambda p}}{k!} \end{aligned}$$

So $D \sim \text{Poisson}(\lambda p)$. If there is only one parking space, the probability you will be able to park is $P(D = 0) = e^{-\lambda p}$ since you can only park if no driver wish to park.