

STAT 311 Old midterm problems and answers

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1. A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean and the variance of the number of trials if unsuccessful keys are eliminated.

solution. Let X be the number of trials. Then $P(X = 1) = \frac{1}{n}$, $P(X = 2) = \frac{n-1}{n} \frac{1}{n-1} = \frac{1}{n}$, $\dots$, $P(X = n) = \frac{n-1}{n} \dots \frac{1}{2} = \frac{1}{n}$. The random variable X a discrete uniform random variable in integer values $\{1, 2, \dots, n\}$.

$$\mathbb{E}X = \sum_{x=1}^n \frac{x}{n} = \frac{1}{n} \sum_{x=1}^n x = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

$$\mathbb{E}X^2 = \sum_{x=1}^n \frac{x^2}{n} = \frac{1}{n} \frac{n(n+1)(2n+1)}{6}.$$

$$\text{So } \mathbf{Var}X = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}.$$

2. Let X have the probability density function $f(x) = ce^{-x}$ for $1 < x < \infty$ and $f(x) = 0$ otherwise. (a) Find the value c . (b) Find the cumulative distribution function of X and plot its graph. (c) Find $P(-1 < X \leq 3 | X \geq 3)$.

Solution. $\int_1^\infty ce^{-x} dx = -ce^{-x}|_1^\infty = ce^{-1} = 1$. So $c = e$ and $f(x) = ee^{-x} = e^{1-x}$ for $1 < x < \infty$. Integrating f , we get $F(x) = 0$ for $x \leq 1$ and $F(x) = 1 - e^{1-x}$ for $x > 1$. F is a nondecreasing function and $F(\infty) = 1$. $P(-1 < X \leq 3 | X \geq 3) = \frac{P(X=3)}{P(X \geq 3)} = 0$ since the probability that a continuous random variable taking a single point is zero.

3. Suppose X has a uniform distribution over $(-1, 1)$ and $Y = \frac{1}{X^2}$. Find the formula for cumulative distribution function of Y in terms of $F_X(x)$ only, where $F_X(x)$ is the cumulative distribution function of X . Find the density function of Y .

solution. $F_Y(y) = P(\frac{1}{y} \leq X^2) = P(\frac{1}{\sqrt{y}} \leq X) + P(X < -\frac{1}{\sqrt{y}}) = 1 - F_X(\frac{1}{\sqrt{y}}) + F_X(-\frac{1}{\sqrt{y}})$ for $y > 1$ and $F_Y(y) = 0$ otherwise. Differentiating with respect to y , $f_Y(y) = \frac{1}{2y^{3/2}} \left(f_X(\frac{1}{\sqrt{y}}) + f_X(-\frac{1}{\sqrt{y}}) \right) = \frac{1}{2y^{3/2}}$ for $y > 1$.

4. (a) The life time X of an electronic component follows the distribution $f(x) = e^{-x}$ for $x > 0$ and $f(x) = 0$ otherwise. Three of these components operate independently in a piece of equipment. The equipment fails if at least two of the components fail. Find the probability that the equipment will operate for at least 2 hours without failure.

Solution. The probability of failure for a single component is

$$p = P(X < 2) = \int_0^2 e^{-x} dx = 1 - e^{-2}.$$

The equipment will fail if at least two out of 3 components fail. So the probability of the equipment failure is

$$\binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3 = 3p^2 (1-p) + p^3.$$

The probability of the equipment operating for at least 2 hours is then $1 - 3p^2(1-p) - p^3$.

- (b) The achievement scores for a college entrance examination are normally distributed with mean 70 and variance 10. What fraction of the scores lies between 75 and 80? Hint: The standard normal table is provided.

Solution. Let X be the achievement score and let $Z = \frac{X-70}{\sqrt{10}} \sim N(0, 1)$. Need to find

$$p = P\left(\frac{75-70}{\sqrt{10}} < Z < \frac{80-70}{\sqrt{10}}\right) = P(0.5 < Z < 1).$$

Then the probability is $p = \Phi(1) - \Phi(0.5) = 0.15$.